

EXPLICIT CALCULATION OF SIU'S EFFECTIVE TERMINATION IN KOHN'S ALGORITHM FOR SPECIAL DOMAINS IN \mathbb{C}^3

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ABSTRACT. In this article, we follow the arguments in a paper of Y-T. Siu to study the effective termination of Kohn's algorithm for special domains in \mathbb{C}^3 . We make explicit the effective constants and generic conditions that appear there, and we obtain an explicit expression for the regularity of the Dolbeault laplacian for the $\bar{\partial}$ -Neumann problem. Specifically, on a local pseudoconvex domain of the special shape

$$\Omega := \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : 2\operatorname{Re} z_3 + \sum_{i=1}^N |F_i(z_1, z_2)|^2 < 0 \right\}$$

with holomorphic function germs $F_1, \dots, F_N \in \mathcal{O}_{\mathbb{C}^2, 0}$ of finite intersection multiplicity

$$s := \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2, 0} / \langle F_1, \dots, F_N \rangle < \infty,$$

we show that an ε -subelliptic regularity for $(0, 1)$ -forms holds whenever, just in terms of s ,

$$\varepsilon \geq \frac{1}{2^{(4s^2-1)s+3} s^2 (4s^2-1)^4 \binom{8s+1}{8s-1}}.$$

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1. INTRODUCTION

Let (z_1, z_2, z_3) be holomorphic coordinates in \mathbb{C}^3 . For some $N \geq 1$, let $F_1(z_1, z_2), \dots, F_N(z_1, z_2)$ be holomorphic function germs of \mathbb{C}^3 vanishing at the origin. We shall study the ideal of subelliptic multipliers on special domains $\Omega \subseteq \mathbb{C}^3$ defined by

$$\Omega = \left\{ 2\operatorname{Re} z_3 + \sum_{i=1}^N |F_i(z_1, z_2)|^2 < 0 \right\}.$$

The idea of subelliptic multipliers was conceived by Joseph J. Kohn in [Koh79] to study the $\bar{\partial}$ -Neumann problem on pseudoconvex domains in \mathbb{C}^n . He constructed what is now known as the *Kohn's algorithm* on subelliptic multipliers to give a geometric interpretation of the subelliptic estimates of the Dolbeault Laplacian. With the use of Diederich–Fornaess' theorem [DF78], he proved that the termination of Kohn's algorithm is equivalent to the absence of holomorphic curves passing through the origin in $b\Omega$.

For special domains in \mathbb{C}^2 , this is equivalent to the fact that in a neighbourhood of the origin, the intersection of the variety germs $\cap_{i=1}^N \{(z_1, z_2) : F_i(z_1, z_2) = 0\}$ consists only of the origin. By a result in intersection theory, this means that

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \langle F_1, \dots, F_N \rangle := s < \infty.$$

An important problem with the termination of Kohn's algorithm is its *effectiveness*. Throughout this paper, we shall say that a certain quantity is *effective* if it can be expressed in terms of s . In [CD10], John P. D'Angelo and David W. Catlin proved the effective termination of the algorithm for triangular systems, and raised an example where the termination fails to be effective.

In [Siu10], Yum-Tong Siu proved the effective termination of Kohn's algorithm from the view of local intersection theory in several complex variables to create multipliers with effective multiplicities. Based on his method, both the number of steps taken to terminate the algorithm, and the regularity of the Dolbeault laplacian are effective for special domains in \mathbb{C}^n . Here, we will follow the exposition in [Siu10] for the case of dimension 3.

Let us briefly outline Siu's method. The first step of Kohn's algorithm allows only a linear combination of the F_i . One idea is to create *generic* linear combinations

$$A = \sum_{i=1}^N \lambda_i F_i, \quad B = \sum_{i=1}^N \mu_i F_i$$

whose intersection multiplicity

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \langle A, B \rangle$$

has an effective upper bound. The next step in Kohn's algorithm consists of taking the Jacobian $\operatorname{Jac}(A, B)$, and of letting \tilde{h}_2 be the reduction of $\operatorname{Jac}(A, B)$. The holomorphic function \tilde{h}_2 has an effective multiplicity. Furthermore, there exists another generic linear combination $h_1 := \sum_{i=1}^N c_i F_i$

such that

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \langle \tilde{h}_2, h_1 \rangle$$

has an effective upper bound. From h_1 and \tilde{h}_2 , one may construct a holomorphic function $h_2(z_1, z_2)$ with an effective vanishing order λ for $z_2 \mapsto h_2(0, z_2)$ so that $h_2(h_1(z_1, z_2), z_2)$ is a subelliptic multiplier, hence up to a multiplication by a unit, $h_2(h_1, z_2)$ may be written as a Weierstrass polynomial

$$h_2(h_1, z_2) = z_2^\lambda + \sum_{1 \leq j \leq \lambda} a_j(h_1) z_2^{\lambda-j}.$$

The rest of the argument consists of applying Kohn's algorithm to the pre-multiplier h_1 together with the subelliptic multiplier $h_2(h_1, z_2)$. The algorithm terminates with an effective number of steps, because $h_2(h_1, z_2)$ is a polynomial with an effective degree λ .

The purpose of this paper is to review some of the useful concepts in several complex variables, and to describe in greater detail the generic conditions wherever they appear. Then we will make explicit the effective constants and upper bounds that are found during the course of creating subelliptic multipliers, and we will apply these results to explicitly describe the regularity of the Dolbeault laplacian. The following is our main result:

Theorem 1.1. *Let (z_1, z_2, z_3) be holomorphic coordinates in \mathbb{C}^3 with $z_i = x_i + \sqrt{-1}y_i$. For some $N \geq 2$, let $F_1(z_1, z_2), \dots, F_N(z_1, z_2)$ be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^2,0}$ vanishing at the origin such that*

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \langle F_1, \dots, F_N \rangle := s < \infty.$$

Let $\Omega \subset \mathbb{C}^3$ be the domain defined by

$$\Omega = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : 2\operatorname{Re} z_3 + \sum_{i=1}^N |F_i(z_1, z_2)|^2 < 0 \right\}.$$

Then by Siu's method, Kohn's algorithm terminates in at most $(4s^2 - 1)s$ steps. Moreover, for any $\phi \in \mathcal{D}_{0,1}(\Omega)$ with compact support,

$$\|\phi\|_{\varepsilon}^2 \lesssim \|\bar{\partial}\phi\|^2 + \|\bar{\partial}^* \phi\|^2 + \|\phi\|^2,$$

where

$$\varepsilon \geq \frac{1}{2^{(4s^2-1)s+3}s^2(4s^2-1)^4 \binom{8s+1}{8s-1}}.$$

(See the next section for the definitions of $\mathcal{D}_{0,1}(\Omega)$ and of the tangential Sobolev norm $\|\cdot\|_{\varepsilon}^2$.)

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2. THE $\bar{\partial}$ -NEUMANN PROBLEM

Let $\Omega \subseteq \mathbb{C}^n$ be an open domain in \mathbb{C}^n , and let $\mathcal{E}^{p,q}(\Omega)$ denote the set of smooth (p, q) -forms on Ω . More explicitly, every element $\phi \in \mathcal{E}^{p,q}(\Omega)$ can be written in the form

$$\phi = \sum_{\substack{(i_1, \dots, i_p) \in \mathbb{N}^p, \\ 1 \leq i_1 < \dots < i_p \leq n}} \sum_{\substack{(j_1, \dots, j_q) \in \mathbb{N}^q, \\ 1 \leq j_1 < \dots < j_q \leq n}} \phi_{i_1, \dots, i_p, j_1, \dots, j_q} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q},$$

where $\phi_{i_1, \dots, i_p, j_1, \dots, j_q} \in \mathcal{E}^{0,0}(\Omega) = C^\infty(\Omega)$. For notational convenience, ϕ may be written as

$$(2.1) \quad \phi = \sum'_{|I|=p} \sum'_{|J|=q} \phi_{IJ} dz_I \wedge d\bar{z}_J.$$

The notation \sum' denotes the sum over increasing indices.

Definition 2.2. Let $\mathcal{E}^{p,q}(\bar{\Omega})$ denote the following subset of $\mathcal{E}^{p,q}(\Omega)$:

$$\mathcal{E}^{p,q}(\bar{\Omega}) := \left\{ \phi \in \mathcal{E}^{p,q}(\Omega) : \text{there exists a neighbourhood } V \text{ of } \bar{\Omega} \text{ and a smooth } \tilde{\phi} \in \mathcal{E}^{p,q}(V) \text{ such that } \tilde{\phi}|_V = \phi \right\}.$$

For any f, g in $L^2_{(p,q)}(\Omega)$

$$f = \sum'_{|I|=p} \sum'_{|J|=q} f_{IJ} dz_I \wedge d\bar{z}_J, \quad \text{and} \quad g = \sum'_{|I|=p} \sum'_{|J|=q} g_{IJ} dz_I \wedge d\bar{z}_J$$

with

$$\sum'_{|I|=p} \sum'_{|J|=q} \int_{\Omega} |f_{IJ}|^2 dV < \infty \quad \text{and} \quad \sum'_{|I|=p} \sum'_{|J|=q} \int_{\Omega} |g_{IJ}|^2 dV < \infty,$$

the metric $(-, -)$ on $L^2_{(p,q)}(\Omega)$ is defined by

$$(f, g) := \sum'_{|I|=p} \sum'_{|J|=q} \int_{\Omega} f_{IJ} \overline{g_{IJ}} d\lambda.$$

Here $d\lambda$ denotes the Lebesgue measure on \mathbb{C}^n .

2.1. The $\bar{\partial}$ Operator. Let $\phi \in \mathcal{E}^{p,q}(\Omega)$ as in equation (2.1). The differential operator $\bar{\partial}$ is then a map $\bar{\partial} : \mathcal{E}^{p,q}(\Omega) \rightarrow \mathcal{E}^{p,q+1}(\Omega)$ defined by

$$\begin{aligned} \bar{\partial}\phi &= \bar{\partial} \left(\sum'_{|I|=p} \sum'_{|J|=q} \phi_{IJ} dz_I \wedge d\bar{z}_J \right) \\ &= \sum'_{|I|=p} \sum'_{|J|=q} \sum_{j=1}^n \frac{\partial \phi_{IJ}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J. \end{aligned}$$

Definition 2.3. Let X and Y be Banach spaces. An *unbounded operator* T on X with target in Y is consists of a linear subspace $\text{Dom}(T)$ called the *domain of T* , and a linear map

$$T : \text{Dom}(T) \rightarrow Y.$$

The unbounded operator T will be written as

$$(T, \text{Dom}(T)) : X \rightarrow Y.$$

Definition 2.4. Let X and Y be Banach Spaces. An unbounded operator $(T, \text{Dom}(T)) : X \rightarrow Y$ is *closed* if the graph of T is closed.

Definition 2.5. Let X and Y be Banach spaces, and let

$$(T, \text{Dom}(T)) : X \rightarrow Y$$

be an unbounded operator. Then the unbounded operator T is *densely defined* if $\text{Dom}(T)$ is dense in X .

Note that even if $\phi \in L^2_{p,q}(\Omega)$, one may still define the $(p, q+1)$ form $\bar{\partial}\phi$ in the sense of currents. The space of $(p, q+1)$ -currents contains the space $L^2_{p,q+1}(\Omega)$.

Definition 2.6. Let $\bar{\partial}$ be the operator as above. Then $\text{Dom}_{p,q}(\bar{\partial})$ denotes the following linear subspace of $L^2_{p,q}(\Omega)$:

$$\text{Dom}_{p,q}(\bar{\partial}) := \{\phi \in L^2_{p,q}(\Omega) : \bar{\partial}\phi \in L^2_{p,q+1}(\Omega)\}.$$

Clearly since $\text{Dom}_{p,q}(\bar{\partial})$ contains the space of all (p, q) forms on Ω with compact support, which forms a dense set in $L^2_{p,q}(\Omega)$, hence $\text{Dom}_{p,q}(\bar{\partial})$ is a dense set.

The pair

$$(\bar{\partial}, \text{Dom}_{p,q}(\bar{\partial})) : L^2_{p,q}(\Omega) \rightarrow L^2_{p,q+1}(\Omega)$$

defines a closed, densely defined unbounded operator.

2.2. The Hilbert Space adjoint of $\bar{\partial}$. Before we define the Hilbert space adjoint $\bar{\partial}^*$ of $\bar{\partial}$, we first specify the domain of $\bar{\partial}^*$.

Definition 2.7 ($\text{Dom}_{p,q}(\bar{\partial}^*)$). Let $\text{Dom}_{p,q}(\bar{\partial}^*)$ be the following linear subspace of $L^2_{p,q}(\Omega)$:

$$\begin{aligned} & \text{Dom}_{p,q}(\bar{\partial}^*) \\ &:= \left\{ \phi \in L^2_{p,q}(\Omega) : \text{the map } T_\phi : \text{Dom}_{p,q}(\bar{\partial}^*) \rightarrow \mathbb{C} \text{ defined by } T_\phi(u) = (\phi, \bar{\partial}u) \text{ is continuous} \right\}. \end{aligned}$$

From the definition of the domain $\text{Dom}_{p,q}(\bar{\partial}^*)$, the action of $\bar{\partial}^*$ on $\text{Dom}_{p,q}(\bar{\partial}^*)$ may be defined as follows: let T_ϕ be the map in the definition. If $\phi \in \text{Dom}_{p,q}(\bar{\partial}^*)$, then linear map

$$\begin{aligned} T_\phi : \text{Dom}_{p,q}(\bar{\partial}^*) &\rightarrow \mathbb{C} \\ u &\mapsto (\phi, \bar{\partial}u) \end{aligned}$$

is continuous on the subspace $\text{Dom}_{p,q}(\Omega) \subseteq L^2_{p,q}(\Omega)$. By the Hahn-Banach theorem, there exists an extension

$$\tilde{T}_\phi : L^2_{p,q}(\Omega) \rightarrow L^2_{p,q}(\Omega)$$

of T_ϕ to the whole of Hilbert space $L^2_{p,q}(\Omega)$. This extension is unique since $\text{Dom}_{p,q}(\bar{\partial}^*)$ is dense. Also, \tilde{T}_ϕ is a continuous linear operator. By Riesz representation theorem, there exists the unique element $\bar{\partial}^*\phi$ such that for all $u \in L^2_{p,q}(\Omega)$,

$$\tilde{T}_\phi(u) = (\bar{\partial}^*\phi, u).$$

If $u \in \text{Dom}_{p,q}(\bar{\partial})$, then

$$(\bar{\partial}^*\phi, u) = \tilde{T}_\phi(u) = T_\phi(u) = (\phi, \bar{\partial}u).$$

Definition 2.8 (Hilbert space adjoint of $\bar{\partial}$). The Hilbert space adjoint $\bar{\partial}^*$ of $\bar{\partial}$ is an unbounded operator

$$(\bar{\partial}^*, \text{Dom}_{p,q}\bar{\partial}^*) : L_{p,q}^2(\Omega) \rightarrow L_{p,q-1}^2(\Omega)$$

such that for all $\phi \in \text{Dom}_{p,q}(\bar{\partial}^*)$ and $u \in \text{Dom}_{p,q-1}(\bar{\partial})$,

$$(\bar{\partial}^* \phi, u) = (\phi, \bar{\partial} u).$$

2.3. The Dolbeault Laplacian $\Delta_{\bar{\partial}}$. The following unbounded operators $\bar{\partial}$ and $\bar{\partial}^*$ act in the following way

$$L_{p,q-1}^2(\Omega) \xrightleftharpoons[\bar{\partial}^*]{\bar{\partial}} L_{p,q}^2(\Omega) \xrightleftharpoons[\bar{\partial}^*]{\bar{\partial}} L_{p,q+1}^2(\Omega).$$

As a result, there is an unbounded operator

$$(\Delta_{\bar{\partial}}, \text{Dom}_{p,q}(\Delta_{\bar{\partial}})) : L_{p,q}^2(\Omega) \rightarrow L_{p,q}^2(\Omega)$$

called the Dolbeault laplacian

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial},$$

defined on

$$\begin{aligned} & \text{Dom}_{p,q}(\Delta_{\bar{\partial}}) \\ &= \left\{ \phi \in L_{p,q}^2(\Omega) : \phi \in \text{Dom}_{p,q}(\bar{\partial}) \cap \text{Dom}_{p,q}(\bar{\partial}^*), \bar{\partial}\phi \in \text{Dom}_{p,q+1}(\bar{\partial}^*), \bar{\partial}^*\phi \in \text{Dom}_{p,q-1}(\bar{\partial}) \right\}. \end{aligned}$$

2.4. The Subelliptic estimate and Subelliptic multipliers.

Definition 2.9 ($\mathcal{D}_{p,q}(\Omega)$). The set $\mathcal{D}_{p,q}(\Omega)$ is defined to be

$$\mathcal{D}_{p,q}(\Omega) = \text{Dom}_{p,q}(\bar{\partial}^*) \cap \mathcal{E}^{p,q}(\bar{\Omega}).$$

Definition 2.10 (The Tangential Sobolev Norm). Let $f(t_1, \dots, t_{2n-1}, r) \in \mathcal{S}(\mathbb{R}^{2n-1} \times \mathbb{R})$. The pseudodifferential operator of order s , denoted by Λ^s , is defined by

$$\Lambda^s f = \frac{1}{(2\pi)^{\frac{2n-1}{2}}} \int_{r=-\infty}^0 \int_{\mathbb{R}^{2n-1}} e^{\sqrt{-1} \sum_{k=1}^{2n-1} t_k \tau_k} \left(1 + \sum_{k=1}^{2n-1} |\tau_k|^2 \right)^{s/2} \hat{f}(\tau_1, \dots, \tau_{2n-1}, r) d\tau dr,$$

where

$$\hat{f}(\tau_1, \dots, \tau_{2n-1}, r) = \frac{1}{(2\pi)^{\frac{2n-1}{2}}} \int_{\mathbb{R}^{2n-1}} e^{-\sqrt{-1} \sum_{k=1}^{2n-1} t_k \tau_k} f(t_1, \dots, t_{2n-1}, r) dt.$$

The tangential sobolev norm $\|\bullet\|_s^2$ is defined by

$$\|f\|_s^2 = \frac{1}{(2\pi)^{\frac{2n-1}{2}}} \int_{r=-\infty}^0 \int_{\mathbb{R}^{2n-1}} \left(1 + \sum_{k=1}^{2n-1} |\tau_k|^2 \right)^{\frac{s}{2}} |\hat{f}(\tau_1, \dots, \tau_{2n-1}, r)|^2 d\tau dr.$$

By [FK72, Appendix, Proposition A.3.1], if $s > s'$, then for any $\varepsilon > 0$, there exists a neighbourhood $V \subset \mathbb{R}^{2n}$ of the origin such that $\|u\|_{s'} \leq \varepsilon \|u\|_s$ for all u supported in V .

Definition 2.11 (The Subelliptic Estimates). Suppose that $\Omega \subset\subset \mathbb{C}^n$ is an open domain whose closure is compact, and whose boundary is smooth. Let $x \in \overline{\Omega}$. The $\bar{\partial}$ -Neumann problem satisfies a subelliptic estimate on $(0, q)$ forms if there exists a neighbourhood $U \subseteq \mathbb{C}^n$ of x , and positive constants c, ε , such that for all $\phi \in \mathcal{D}_{0,q}(U \cap \Omega)$ with compact support,

$$\|\phi\|_\varepsilon^2 \leq c \left(\|\bar{\partial}\phi\|^2 + \|\bar{\partial}^*\phi\|^2 + \|\phi\|^2 \right).$$

From here, we will adopt the following notation: we let $Q(\phi, \psi)$ denote the quadratic form

$$Q(\phi, \psi) = (\bar{\partial}\phi, \bar{\partial}\psi) + (\bar{\partial}^*\phi, \bar{\partial}^*\psi) + (\phi, \psi).$$

Definition 2.12 (Subelliptic multipliers). Let Ω be a smoothly bounded pseudoconvex domain in \mathbb{C}^n . Let $x \in \overline{\Omega}$ be a point, and let \mathcal{C}_x^∞ be the ring of germs of smooth functions at x . An element $g \in \mathcal{C}_x^\infty$ is a subelliptic multiplier on $(0, 1)$ forms if there exists a neighbourhood $U \subseteq \mathbb{C}^n$ of x , and positive constants c, ε , such that for all $\phi \in \mathcal{D}_{0,q}(U \cap \Omega)$ with compact support,

$$\|g\phi\|_\varepsilon^2 \leq cQ(\phi, \phi).$$

3. KOHN'S ALGORITHM FOR SUBELLIPTIC MULTIPLIERS

Let $(z_1, \dots, z_n, x_{n+1} + \sqrt{-1}y_{n+1})$ be a holomorphic coordinates of \mathbb{C}^{n+1} . Let F_1, \dots, F_N be holomorphic function germs vanishing at the origin in \mathbb{C}^{n+1} . For convenience, we let $z := (z_1, \dots, z_n)$. Let $r(z, z_{n+1}, \bar{z}, \bar{z}_{n+1})$ be the real analytic function defined by

$$r(z, z_{n+1}, \bar{z}, \bar{z}_{n+1}) = 2\operatorname{Re}(z_{n+1}) + \sum_{j=1}^N |F_j(z)|^2 = x_{n+1} + \sum_{j=1}^N |F_j(z)|^2.$$

Let Ω be the open domain defined by

$$\Omega = \{r < 0\},$$

and the boundary $b\Omega$ is the following set

$$b\Omega = \{r = 0\}$$

which is smooth. Clearly, $0 \in b\Omega$.

Definition 3.1 (Kohn's Algorithm for Special Domains). Let $\mathcal{I}_0 := \langle F_1, \dots, F_N \rangle$ be the ideal in $\mathcal{O}_{\mathbb{C}^n, 0}$ generated by the holomorphic function germs F_i 's. We associate with \mathcal{I}_0 with a sequence of radical ideals

$$\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \dots$$

in $\mathcal{O}_{\mathbb{C}^n, 0}$ as follows:

(i) Let g_1, \dots, g_n be linear combinations of the F_i

$$g_i = \sum_{k=1}^N c_{ik} F_k \quad (c_{ik} \in \mathbb{C}).$$

Let $\operatorname{Jac}(g_1, \dots, g_n)$ be the Jacobian of the g_i and define

$$\mathcal{I}_1^\# = \langle \operatorname{Jac}(g_1, \dots, g_n) : g_i \text{ is a linear combination of } F_i \rangle.$$

Then set $\mathcal{I}_1 = \sqrt{\mathcal{I}_1^\#}$.

(ii) (Inductive Step) Suppose that \mathcal{I}_k has been constructed, let $\mathcal{I}_{k+1, \text{Jac}}^\#$ be the ideal generated by $\text{Jac}(h_1, \dots, h_n)$ where each h_i is either an element of \mathcal{I}_k , or is a linear combination g_k of the F_i . Then let

$$\mathcal{I}_{k+1}^\# = \mathcal{I}_{k+1, \text{Jac}}^\# + \mathcal{I}_k,$$

and set $\mathcal{I}_{k+1} = \sqrt{\mathcal{I}_{k+1}^\#}$.

Here are some effects of Kohn's algorithm on ε the subelliptic regularity of the multipliers.

Proposition 3.2. *Let $f \in \mathcal{O}_{\mathbb{C}^n, 0}$ be a subelliptic multiplier and let $g \in \mathcal{O}_{\mathbb{C}^n, 0}$ be a holomorphic function germ. This means that at $0 \in b\Omega$, there exists an open neighbourhood $U \subseteq \mathbb{C}^n$ of 0, and strictly positive constants c, ε such that for all $\phi \in \mathcal{D}_{0,1}(U \cap \Omega)$, one has*

$$\|f\phi\|_\varepsilon^2 \leq cQ(\phi, \phi).$$

(i) *Suppose there exists $N > 0$ such that $|g|^N \leq |f|$, then g is also a subelliptic multiplier and*

$$\|g\phi\|_{\varepsilon/N}^2 \leq cQ(\phi, \phi).$$

(ii) *If the $(0, 1)$ -form ϕ is written as $\phi = \sum_{k=1}^n \phi_k d\bar{z}_i$, one has*

$$\left\| \sum_{k=1}^n \frac{\partial f}{\partial z_k} \phi_k \right\|_{\varepsilon/2}^2 \leq cQ(\phi, \phi).$$

(iii) *Let f_1, \dots, f_n be subelliptic multipliers. Suppose there exists $\varepsilon > 0$ such that for all i ,*

$$\left\| \sum_{k=1}^n \frac{\partial f_i}{\partial z_k} \phi_k \right\|_\varepsilon^2 \leq cQ(\phi, \phi),$$

then

$$\|\text{Jac}(f_1, \dots, f_n)\phi\|_\varepsilon^2 \leq cQ(\phi, \phi).$$

(iv) *For any $g \in \mathcal{O}_{\mathbb{C}^n, 0}$, one has*

$$\|gf\phi\|_\varepsilon^2 \lesssim Q(\phi, \phi).$$

Proof. For properties (i) to (iii), see [D'A93]. For the last property, we may refer to [Koh79, p. 94, Proposition 4.7(D)]¹. We will give a summary of the proof of the last property. Given $f \in \mathcal{O}_{\mathbb{C}^n}(U)$ for some open neighbourhood $U \subseteq \mathbb{C}^n$ of the origin, there exists $\varepsilon > 0$ such that for all $\phi \in \mathcal{D}_{0,1}(U \cap \Omega)$, one has

$$\|f\phi\|_\varepsilon^2 \lesssim Q(\phi, \phi).$$

By remark in [Koh79, p 93, Section 4, Paragraph 2], for any $V \subseteq U$ a open subset of U , the same $\varepsilon > 0$ will satisfy the property that for all $\phi \in \mathcal{D}_{0,1}(V \cap \Omega)$,

$$\|f|_{V \cap \bar{\Omega}} \phi\|_\varepsilon^2 \lesssim Q(\phi, \phi).$$

Given $g \in \mathcal{O}_{\mathbb{C}^n, 0}$, for some $V \subseteq U$, $gf \in \mathcal{O}_{\mathbb{C}^n}(V)$. Upon restriction to a smaller open set, g is bounded on \bar{V} . For any $\phi \in \mathcal{D}_{0,1}(V \cap \Omega)$, its support is contained in $V \cap \bar{\Omega}$. Hence

$$\|gf\phi\|_\varepsilon^2 \lesssim \|f\phi\|_\varepsilon^2 \lesssim Q(\phi, \phi). \quad \square$$

¹For proof, see page 97

4. LOCAL GEOMETRY OF COMPLEX SPACES AND LOCAL INTERSECTION THEORY

4.0.1. Throughout this section, we will study the geometry of analytic varieties near the origin.

4.0.2. Let $\mathcal{O}_{\mathbb{C}^n,0}$ denote the ring of holomorphic function germs at the origin. It can be canonically identified with $\mathbb{C}\{z_1, \dots, z_n\}$ the ring of convergent power series.

4.0.3. The ring $\mathcal{O}_{\mathbb{C}^n,0}$ is local and let \mathfrak{m} denote its unique maximal ideal, which can be characterised by one of the following equivalent properties:

(i)

$$\mathfrak{m} := \{f \in \mathcal{O}_{\mathbb{C}^n,0} : f(0) = 0\};$$

(ii)

$$\mathfrak{m} = \langle z_1, \dots, z_n \rangle;$$

(iii) every holomorphic function germ f may be written as $f = \sum_{k \geq 1} f_k$ a sum of homogeneous polynomials f_k of degree k .

4.0.4. For each $l \in \mathbb{N}$, we define \mathfrak{m}^{l+1} recursively by

$$\begin{aligned} \mathfrak{m}^{l+1} &= \mathfrak{m} \cdot \mathfrak{m}^l \\ &= \left\{ \sum_{k=1}^N f_k g_k : N \in \mathbb{N}, f_k \in \mathfrak{m}, g_k \in \mathfrak{m}^l \right\}. \end{aligned}$$

For any fixed $l \geq 1$, the following conditions are equivalent:

(i) $h \in \mathfrak{m}^l$;

(ii) for each $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ such that $\alpha_1 + \dots + \alpha_n \leq l - 1$,

$$(\partial_{z_1}^{\alpha_1} \dots \partial_{z_n}^{\alpha_n} h)(0) = 0;$$

(iii)

$$\mathfrak{m}^l = \langle z_1^{k_1} \dots z_n^{k_n} : k_1 + \dots + k_n = l \rangle;$$

(iv) $h = \sum_{k \geq l} h_k$ where either h_k vanishes or is a homogeneous polynomial of degree k .

4.0.5. *Multiplicity.*

Definition 4.1. Let $h \in \mathcal{O}_{\mathbb{C}^n,0}$ be a holomorphic function germ, which can be written as

$$h = \sum_{k=0}^{\infty} h_k$$

a sum of homogeneous polynomials h_k of degree k . The multiplicity of h , which will be denoted by $\text{mult}_0 h$, is the smallest positive integer k for which $h_k \not\equiv 0$.

4.0.6. By Paragraph 4.0.4((i) \iff (iv)), the holomorphic function h lies in \mathfrak{m}^l if and only if $\text{mult}_0 h \geq l$. In section 5, we will study the geometric characterisation of multiplicity of a holomorphic function, and the extension of this notion to certain ideals.

4.1. Local Analytic Geometry.

4.1.1. In this subsection, we let F_1, \dots, F_N be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n, 0}$ vanishing at the origin. For easier exposition, we will not specify the domain of definition of the holomorphic function germs.

4.1.2. *Local Analytic Set, Germs of Analytic Space.*

Definition 4.2. A set $X \subseteq \mathbb{C}^n$ is *locally analytic* if for any point $p \in X$, there exists an open subset V of p in \mathbb{C}^n , and finitely many holomorphic functions f_1, \dots, f_s defined on V , such that

$$X \cap V = \{x \in V : f_1(x) = \dots = f_s(x) = 0\}.$$

Definition 4.3. A *germ of analytic space* $(X, 0)$ is a germ at 0 of a locally analytic subset of \mathbb{C}^n .

4.1.3. Any germ of an analytic space $(X, 0)$ may be uniquely written as

$$(X, 0) = (X_1, 0) \cup \dots \cup (X_r, 0)$$

a union of irreducible components², each of which is also a germ of an analytic space ([dJP00, Corollary 3.4.18, p 118]).

4.1.4. $(V(F), 0)$, $(V(\mathcal{I}_F), 0)$ and $\mathcal{I}(X, 0)$.

Definition 4.4. Let $F \in \mathcal{O}_{\mathbb{C}^n, 0}$. The germ of an analytic hypersurface $(V(F), 0)$ is defined as follows. Let U be an open neighbourhood of the origin on which F seen as a power series converges. Consider $V(F) = \{p \in U : F(p) = 0\}$. Then $(V(F), 0)$ is the germ of $V(F)$ at zero, and is called the *zero set* of F .

Definition 4.5. Let $\mathcal{I}_F = \langle F_1, \dots, F_N \rangle$ be an ideal of $\mathcal{O}_{\mathbb{C}^n, 0}$. The germ of analytic space $(V(\mathcal{I}_F), 0)$ is defined by

$$(V(\mathcal{I}_F), 0) = \bigcap_{i=1}^N (V(F_i), 0).$$

Definition 4.6. Let $(X, 0)$ be a germ of an analytic space. Then define

$$\mathcal{I}(X, 0) = \{f \in \mathcal{O}_{\mathbb{C}^n, 0} : (X, 0) \subseteq (V(f), 0)\}.$$

4.1.5. *Properties of $(V(\mathcal{I}_F), 0)$ and $\mathcal{I}(X, 0)$.* Let F_1, \dots, F_N and G_1, \dots, G_M be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n, 0}$. Let $\mathcal{I}_F = \langle F_1, \dots, F_N \rangle$ and $\mathcal{I}_G = \langle G_1, \dots, G_M \rangle$ be the corresponding ideals they generate. Let $(X, 0)$ and $(Y, 0)$ be germs of analytic spaces.

- (i) $\mathcal{I}_F \subseteq \mathcal{I}_G$ implies that $(V(\mathcal{I}_G), 0) \subseteq (V(\mathcal{I}_F), 0)$;
- (ii) $(X, 0) \subseteq (Y, 0)$ implies that $\mathcal{I}(Y, 0) \subseteq \mathcal{I}(X, 0)$;
- (iii) for any $k \in \mathbb{N}_{\geq 1}$, and for any ideal \mathcal{I}_F , $(V(\mathcal{I}_F^k), 0) = (V(\mathcal{I}_F), 0)$;
- (iv) for any germ of analytic space $(X, 0)$, $(V(\mathcal{I}(X, 0)), 0) = (X, 0)$;
- (v) (Nullstellensatz) $\mathcal{I}(V(\mathcal{I}_F), 0) = \sqrt{\mathcal{I}_F}$.

For ease of notation, let $V(F_1, \dots, F_N)$ or $V(\mathcal{I}_F)$ denote $(V(\mathcal{I}_F), 0)$.

4.2. Local Intersection Theory I.

²A germ of an analytic space (X, x) is irreducible if whenever $(X, x) = (X_1, x) \cup (X_2, x)$ with (X_1, x) and (X_2, x) germs of analytic spaces, either $(X, x) = (X_1, x)$ or $(X, x) = (X_2, x)$.

4.2.1. We begin with the characterisation of complete intersections of germs of analytic varieties at the origin.

Theorem 4.7. *Let F_1, \dots, F_N be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n,0}$ at the origin. The following statements are equivalent.*

- (i) $V(F_1, \dots, F_N) = \{0\}$;
- (ii) *there exists a positive integer $q \geq 1$ such that $\mathfrak{m}^q \subseteq \mathcal{I}_F$;*
- (iii) *the number*

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}_F =: s$$

is finite;

- (iv) *there exists a positive integer p such that locally*

$$|z|^p \lesssim \sum_{i=1}^N |F_i|.$$

Proof. The proof proceeds in the following manner: (i) \implies (ii) \implies (iii) \implies (i), and (ii) \iff (iv).

For (i) \implies (ii), since $V(F_1, \dots, F_N) = \{0\} = V(\mathfrak{m})$, there is an equality of ideals $\mathcal{I}(V(F_1, \dots, F_N)) = \mathcal{I}(V(\mathfrak{m}))$. By Nullstellensatz, therefore $\mathfrak{m} = \sqrt{\mathfrak{m}} = \sqrt{\mathcal{I}_F}$. Hence there exists $q \in \mathbb{N}_{\geq 1}$ such that $\mathfrak{m}^q \subseteq \mathcal{I}_F$.

For (ii) \implies (iii), the condition that $\mathfrak{m}^q \subseteq \mathcal{I}_F$ implies that there is a surjective map of \mathbb{C} -vector space

$$\begin{aligned} \mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{m}^q &\longrightarrow \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}_F \\ f \bmod \mathfrak{m}^q &\longmapsto f \bmod \mathcal{I}_F. \end{aligned}$$

Hence,

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}_F \leq \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{m}^q,$$

and the proof is complete since $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{m}^q$ is always finite for $q \in \mathbb{N}_{\geq 1}$.

For (iii) \implies (i), it is needed to show that the set

$$\{(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n : F_k(\alpha_1, \dots, \alpha_n) = 0 \text{ for all } 1 \leq k \leq N\}$$

is finite. To this effect, it suffices to show that there can only be finitely many choices for each α_i . Since $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}_F$ is finite dimensional, for each $1 \leq i \leq n$, there exists $k_i \in \mathbb{N}_{\geq 1}$ such that the classes

$$\{1, z_i, \dots, z_i^{k_i}\}$$

form a linearly dependent set in $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}_F$. Hence there exist constants $\{c_{i,0}, \dots, c_{i,k_i}\}$ such that

$$\sum_{j=0}^{k_i} c_{ij} z_i^j \equiv 0 \bmod \mathcal{I}_F.$$

Thus there exists a holomorphic function $h_i(z_1, \dots, z_n) \in \mathcal{I}_F$ such that

$$\sum_{j=0}^{k_i} c_{ij} z_i^j - h_i(z_1, \dots, z_n) \equiv 0 \quad (1 \leq i \leq n).$$

If $(\alpha_1, \dots, \alpha_n) \in \mathcal{I}_F$, then for all $1 \leq i \leq n$, one has $h_i(\alpha_1, \dots, \alpha_n) = 0$. Hence

$$\sum_{j=0}^{k_i} c_{ij} \alpha_i^j = 0.$$

The equation above is a polynomial equation in degree k_i , and so there are at most k_i distinct solutions for α_i . This holds for all i , and therefore $V(\mathcal{I}_F)$ is a finite set. The proof is complete.

The implication (ii) \implies (iv) is immediate. The converse will be proved after Skoda's theorem is introduced. The proof is reproduced from [Siu10, p 1179] \square

Theorem 4.8 (Theorem of Henri Skoda). *Let D be a pseudoconvex domain in \mathbb{C}^n and let χ be a plurisubharmonic function on D . Let g_1, \dots, g_m be holomorphic functions on D . Let $\alpha > 1$ and $l = \min\{n, m-1\}$. Then for every holomorphic function F on D such that*

$$\int_D |F|^2 |g|^{-2\alpha l - 2} e^{-\chi} < \infty,$$

there exist holomorphic functions h_1, \dots, h_m on D such that

$$F = \sum_{i=1}^m h_i g_i,$$

and

$$\int_D |h|^2 |g|^{-2\alpha l - 2} e^{-\chi} \leq \frac{\alpha}{\alpha - 1} \int_D |F|^2 |g|^{-2\alpha l - 2} e^{-\chi},$$

where $|g| = (\sum_{i=1}^m |g_i|^2)^{1/2}$ and $|h| = (\sum_{i=1}^m |h_i|^2)^{1/2}$.

Finishing the proof of Theorem. For any non-negative numbers $\gamma_1, \dots, \gamma_n$ with $\gamma_1 + \dots + \gamma_n = (n+2)p$, Skoda's theorem is applied with the following variables: $F = z_1^{\gamma_1} \dots z_n^{\gamma_n}$, $m = N + n$, $\chi \equiv 0$, $(F_1, \dots, F_N, 0, \dots, 0) = (g_1, \dots, g_m)$, $l = n$ and $\alpha = \frac{n+1}{n}$. By the hypothesis in (iv),

$$|z_1^{\gamma_1} \dots z_n^{\gamma_n}|^2 \lesssim |z|^{2(n+2)p} \lesssim \left(\sum_{i=1}^N |F_i| \right)^{2(n+2)} \lesssim \left(\sum_{i=1}^N |F_i|^2 \right)^{(n+2)}$$

where the last inequality follows from Jensen's inequality. Hence over a small pseudoconvex domain D ,

$$\int_D \frac{|z_1^{\gamma_1} \dots z_n^{\gamma_n}|^2}{\left(\sum_{j=1}^2 |F_j|^2 \right)^{n+2}} \lesssim \int_D \frac{\left(\sum_{i=1}^N |F_i|^2 \right)^{(n+2)}}{\left(\sum_{i=1}^N |F_i|^2 \right)^{(n+2)}} = \int_D 1 < \infty.$$

Skoda's theorem applies and therefore $z_1^{\gamma_1} \dots z_n^{\gamma_n} \in \mathcal{I}_F$. Consequently, $\mathfrak{m}^{(n+2)p} \subseteq \mathcal{I}_F$. \square

From the proof above, we obtain the following corollary.

Corollary 4.9. *Let F_1, \dots, F_N be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n, 0}$ at the origin, and suppose there exists $p \in \mathbb{N}_{\geq 1}$ such that*

$$|z|^p \lesssim \sum_{i=1}^N |F_i|$$

in a small neighbourhood 0, then $\mathfrak{m}^{(n+2)p} \subseteq \mathcal{I}_F$.

4.2.2. The intersection invariants (p, q, s) .

Definition 4.10. Let F_1, \dots, F_N be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n,0}$ at the origin. The ideal $\mathcal{I}_F = \langle F_1, \dots, F_N \rangle$ is said to have finite intersection multiplicity with data (p, q, s) if

(i) p is the smallest strictly positive integer satisfying

$$|z|^p \lesssim \sum_{i=1}^N |F_i|;$$

(ii) q is the smallest strictly positive integer satisfying

$$\mathfrak{m}^q \subseteq \mathcal{I}_F;$$

(iii) s is following number below

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \mathcal{I}_F =: s.$$

4.2.3. The relations between the intersection invariants.

Proposition 4.11. Let F_1, \dots, F_N be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n,0}$ at the origin so that the ideal they generate \mathcal{I}_F has finite intersection multiplicity with data (p, q, s) . Then we have the following inequalities:

(i) $q \leq s \leq \binom{n+q-1}{q-1}$,

(ii) $p \leq q \leq (n+2)p$.

Proof. To prove $q \leq s$, it is first observed that $\mathcal{O}_{\mathbb{C}^n,0} / \mathcal{I}_F$ is also a local ring with the maximal ideal $\mathfrak{m} / \mathcal{I}_F$. In the chain of inclusion of vector spaces with

$$\frac{\mathcal{O}_{\mathbb{C}^n,0}}{\mathcal{I}_F} \supseteq \frac{\mathfrak{m}}{\mathcal{I}_F} \supseteq \left(\frac{\mathfrak{m}}{\mathcal{I}_F} \right)^2 \supseteq \dots \supseteq \left(\frac{\mathfrak{m}}{\mathcal{I}_F} \right)^{s+1},$$

since $\mathcal{O}_{\mathbb{C}^n,0} / \mathcal{I}_F$ is an s -dimensional complex vector space, there exists a positive integer $1 \leq k \leq s$ such that

$$\left(\frac{\mathfrak{m}}{\mathcal{I}_F} \right)^k = \left(\frac{\mathfrak{m}}{\mathcal{I}_F} \right)^{k+1} = \left(\frac{\mathfrak{m}}{\mathcal{I}_F} \right) \left(\frac{\mathfrak{m}}{\mathcal{I}_F} \right)^k.$$

By Nakayama's lemma³,

$$\left(\frac{\mathfrak{m}}{\mathcal{I}_F} \right)^k \equiv 0 \quad \text{in } \frac{\mathcal{O}_{\mathbb{C}^n,0}}{\mathcal{I}_F}.$$

Therefore, if g_1, \dots, g_k are elements of \mathfrak{m} in $\mathcal{O}_{\mathbb{C}^n,0}$, then the class $g_1 \cdots g_k$ belongs to $(\mathfrak{m} / \mathcal{I}_F)^k$ which is the zero vector space. Hence the holomorphic function $g_1 \cdots g_k$ lies in \mathcal{I}_F . Since the $g_1 \cdots g_k$ generate \mathfrak{m}^k , the ideal \mathfrak{m}^k is contained in \mathcal{I}_F . By the definition of q , the inequality $q \leq k \leq s$ holds.

Next, for $s \leq \binom{n+q-1}{q-1}$, this follows directly from $\mathfrak{m}^q \subseteq \mathcal{I}_F$.

In the second set of inequalities, to prove $p \leq q$, observe that since $\mathfrak{m}^q \subseteq \mathcal{I}_F$,

$$|z|^q \lesssim \sum_{i=1}^N |F_i|.$$

³The following version of Nakayama's lemma is used: let A be a commutative local ring with 1, and \mathfrak{m} its maximal ideal. For any finitely generated A -module M , if $\mathfrak{m}M = M$, then $M = 0$

Hence, by the definition of p , $p \leq q$.

To prove $q \leq (n+2)p$, it follows from Corollary 4.9. \square

4.2.4. Application of the relations of the invariants.

Lemma 4.12. *Let F_1, \dots, F_N be holomorphic function germs such that the intersection multiplicity of $\mathcal{I}_F = \langle F_1, \dots, F_N \rangle$ is finite with data (p, q, s) . If $h \in \mathcal{O}_{\mathbb{C}^n, 0}$ is a holomorphic function germ with $h(0) = 0$, then $h^s \in \mathcal{I}_F$.*

Proof. Since $h(0) = 0$, the function h lies in \mathfrak{m} . Consequently, $h^s \in \mathfrak{m}^s$. By $q \leq s$, there is an inclusion of ideals $\mathfrak{m}^s \subseteq \mathfrak{m}^q$. Therefore, $h^s \in \mathfrak{m}^s \subseteq \mathfrak{m}^q \subseteq \mathcal{I}_F$. \square

4.3. Local Intersection Theory II.

4.3.1. The case where $N = n = \dim \mathbb{C}^n$ brings another set of equivalent conditions for complete local intersection of n holomorphic function germs $F_1, \dots, F_{N=n}$.

4.3.2.

Theorem 4.13. *Let F_1, \dots, F_n be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n, 0}$ such that $F_i(0) = 0$ for all $1 \leq i \leq n$. The following are equivalent:*

- (i) $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / \langle F_1, \dots, F_n \rangle =: s < \infty$;
- (ii) *the holomorphic map of germs of analytic spaces*

$$\begin{aligned} F : (\mathbb{C}^n, 0) &\longrightarrow (\mathbb{C}^n, 0) \\ (z_1, \dots, z_n) &\longmapsto (F_1, \dots, F_n) \end{aligned}$$

defines a ramified s -sheeted analytic covering;

- (iii) *for each $1 \leq i \leq n$, let ε_i be a small strictly positive number, and Γ be given by*

$$\Gamma = \{z : |F_i| = \varepsilon_i\}.$$

Then the residue map of F at the origin equals to s :

$$\text{Res}_0 F = \int_{\Gamma} \frac{dF_1 \wedge \dots \wedge dF_n}{F_1 \dots F_n} = s.$$

Proof. See [D'A93, p 60], [GH94, p 666-667], [Chi89, p 140, Proposition 1] for discussion. \square

4.3.3. We will show that given Theorem 4.13, one has $\text{mult}_0 \text{Jac}(F) \leq s - 1$.

Theorem 4.14. *Let h be a holomorphic function germ. If $h \in \mathcal{I}_F$, then*

$$\int_{\Gamma} \frac{h dz_1 \wedge \dots \wedge dz_n}{F_1 \dots F_n} = 0.$$

Proof. See [D'A93, p 64]. \square

Corollary 4.15. *Let F_1, \dots, F_n be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n, 0}$ vanishing at the origin, whose varieties they define have complete intersection at the origin. Let F be the map in Theorem 4.13(ii). Then $\text{Jac}(F) \notin \langle F_1, \dots, F_n \rangle$.*

Proof. By Theorem 4.13(iii),

$$0 \neq s = \int_{\Gamma} \frac{dF_1 \wedge \dots \wedge dF_n}{F_1 \dots F_n} = \int_{\Gamma} \frac{\text{Jac}(F) dz_1 \wedge \dots \wedge dz_n}{F_1 \dots F_n}.$$

Hence $\text{Jac}(F) \notin \langle F_1, \dots, F_n \rangle$ by Theorem 4.14. \square

Corollary 4.16. *Let F_1, \dots, F_n be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n,0}$ vanishing at the origin so that the ideal \mathcal{I}_F has finite intersection multiplicity with data (p, q, s) . Then the multiplicity of $\text{Jac}(F)$ at the origin cannot be greater than or equal to s .*

Proof. Suppose otherwise that $\text{mult}_0 \text{Jac}(F) \geq s$, then $\text{Jac}(F) \in \mathfrak{m}^s$. From the inequality $q \leq s$, there is an inclusion of ideals $\mathfrak{m}^s \subseteq \mathfrak{m}^q \subseteq \langle F_1, \dots, F_n \rangle$. Hence $\text{Jac}(F) \in \langle F_1, \dots, F_n \rangle$, which contradicts Corollary 4.15. \square

4.3.4. *Miscellaneous Result.* We will state the following result which will be used later.

Proposition 4.17. *Let f_1, \dots, f_{n-1}, f , and g be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n,0}$ such that*

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \langle f_1, \dots, f_{n-1}, fg \rangle < \infty.$$

Then

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \langle f_1, \dots, f_{n-1}, fg \rangle = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \langle f_1, \dots, f_{n-1}, f \rangle + \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \langle f_1, \dots, f_{n-1}, g \rangle.$$

Proof. See [D'A93, p 60, Theorem 1] \square

5. IDEALS GENERATED BY THE COMPONENTS OF GRADIENT

5.0.1. In this section we shall study the ideals generated by the components of the gradient of a holomorphic function. Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be a holomorphic function germ such that $f(0) = 0$. In a first moment, it will be shown that there exists a positive integer k with

$$f^k \in \left\langle \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right\rangle.$$

In a second moment, more accurately, it will be shown that $k = \dim \mathbb{C}^n = n$ works (optimally) rendering k effective.

5.0.2. *Example.* In 1-dimensional complex analysis, every holomorphic function $f(\zeta)$ with $f(0) = 0$ may be factorised as

$$f(\zeta) = \zeta^k g(\zeta),$$

where $g(0) \neq 0$. A differentiation yields

$$f'(\zeta) = \zeta^{k-1}(kg(\zeta) + \zeta g'(\zeta)),$$

and hence $f \in \langle f' \rangle$. Therefore, $k = 1$ works in this case. In the next few paragraphs we will recall some notions in algebraic geometry.

5.0.3. *Spec, Zariski Topology.* Let A be a commutative ring with 1. We let

$$\text{Spec } A := \{\mathfrak{p} \subset A : \mathfrak{p} \text{ is a prime ideal in } A\}.$$

For every ideal $I \subset A$, set

$$V_A(I) = \{\mathfrak{p} \in \text{Spec } A : I \subseteq \mathfrak{p}\}.$$

The sets $V_A(I)$ are defined as closed sets in $\text{Spec } A$, and the collection

$$\{V_A(I) : I \text{ is an ideal of } A\}$$

defines the *Zariski topology* of $\text{Spec } A$. For principal ideals $\langle f \rangle$, $V_A(\langle f \rangle)$ may be written as $V_A(f)$. Therefore,

$$D_A(f) := \text{Spec } A \setminus V_A(f) = \{\mathfrak{p} \in \text{Spec } A : f \notin \mathfrak{p}\}$$

is open in $\text{Spec } A$. The collection

$$\{D_A(f) : f \in A\}$$

forms a basis for the open set in the Zariski topology. To see this, for any ideal I , one has

$$\text{Spec } A \setminus V_A(I) = \bigcup_{f \in I} D_A(f).$$

5.0.4. For any ideal $I \subset A$, there is a one-to-one correspondance between $\text{Spec } A/I$ and $V_A(I)$. On the other hand, let $f \in A$ and A_f be its localisation. Every element in A_f is a class with representative a/f^k for some $a \in A$ and $k \in \mathbb{N}$. The two representatives a/f^k and b/f^l are equal if there exists $j \geq 0$ such that

$$f^j(af^l - bf^k) = 0 \quad (\text{in } A).$$

It is easily seen that like \mathbb{Q} the quotient numbers of \mathbb{Z} , A_f has a ring structure, and there is a one-to-one correspondance (as sets) between $\text{Spec } A_f$ and $D_A(f)$.

5.0.5. Recall that a commutative ring with 1 is *semi-local* if it has only finitely many maximal ideals. We state the following Artin-Tate theorem.

Theorem 5.1. *Let A be a Noetherian integral domain. Then A is semi-local with $\dim A \leq 1$ if and only if there exists $f \in A$ such that A_f is a field.*

Proof. See [GW10], page 562, Corollary B62. □

5.0.6. Recall that the *Krull dimension* of A is given by

$$\dim A := \sup\{k : \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k\}.$$

Moreover, if A is local, Artin-Tate's theorem may be restated as follows: there exists $f \in A$ such that A_f is a field if and only if $\dim A \leq 1$.

5.0.7. For any germ variety $V(\mathcal{I})$ defined by an ideal $\mathcal{I} \subset \mathcal{O}_{\mathbb{C}^n,0}$, the Krull dimension of $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}$ coincides with the usual intuition of dimension.

5.0.8. To see this, recall that the *Weierstrass dimension* of a germ of complex space (X, x) is the least number k such that there exists a Noether normalisation $\pi^* : \mathcal{O}_{\mathbb{C}^k, x} \hookrightarrow \mathcal{O}_{X, x}$. Both the Weierstrass dimension of (X, x) and the Krull dimension of $\mathcal{O}_{X, x} := \mathcal{O}_{\mathbb{C}^n, x}/\mathcal{I}(X, x)$ coincide ([dJP00, Theorem 4.1.9, pp 131]). The Noether normalisation π^* is uniquely induced by the projection

$$\pi : (X, x) \rightarrow (\mathbb{C}^k, x)$$

of the germ variety (X, x) onto (\mathbb{C}^k, x) with finite fibres. By [dJP00, p 129, Lemma 4.14], $\dim (X, x) = \dim (\mathbb{C}^k, x) = k$. Hence to say that $\dim \mathcal{O}_{X, x} \leq 1$ is to say that either X is of dimension 1 or 0 (but the dimension need not be pure).

5.0.9. The following lemma is a restatement of the Artin-Tate's theorem in more geometric terms.

Lemma 5.2. *Let A be an integral Noetherian ring, and let (0) be a point in $\text{Spec } A$. Then the set $\{(0)\}$ is open in the Zariski topology of $\text{Spec } A$ if and only if $\text{Spec } A$ is a finite set and $\dim A \leq 1$.*

Proof. First, it will be shown that the singleton $\{(0)\}$ is open in $\text{Spec } A$ if and only if there exists $f \in A$ such that A_f is a field. Then secondly, it will be shown that $\text{Spec } A$ is a finite set and $\dim A \leq 1$ if and only if A is semi-local (finitely many maximal ideals) and $\dim A \leq 1$.

For the first assertion, suppose that $\{(0)\}$ is open, then for some index Λ ,

$$\{(0)\} = \bigcup_{i \in \Lambda} D_A(f_i).$$

Therefore, $(0) \in D_A(f_i)$ for some $i \in \Lambda$ and hence

$$\{(0)\} \subseteq D_A(f_i) \subseteq \bigcup_{i \in \Lambda} D_A(f_i) = \{(0)\},$$

so that $\{(0)\} = D_A(f_i)$. By paragraph 5.0.4, this means that the ring A_{f_i} has only (0) as its prime ideal, and so A_{f_i} is a field. Conversely, if there exists $f \in A$ such that A_f is a field, then $\text{Spec } A_f = D_A(f) = \{(0)\}$ as sets. Consequently, the singleton $\{(0)\}$ is open in $\text{Spec } A$.

For the second assertion, suppose that $\text{Spec } A$ is a finite set and $\dim A \leq 1$. The first condition implies that A only has finitely many prime ideals. On the other hand, for any maximal chain of prime ideals

$$(0) := \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k$$

one has $k \leq 1$ by the second condition. If $k = 1$, then \mathfrak{p}_1 is a maximal ideal. If $k = 0$, then \mathfrak{p}_0 is just the zero ideal. Hence any non-zero prime ideal \mathfrak{p}_1 is maximal, and since A has finitely many prime ideals, A is semi-local. Conversely, suppose that A is semi-local and $\dim A \leq 1$. By the chain of inclusion of prime ideals above, any non-zero prime ideal is maximal by second condition. Moreover, A is semi-local means that there are only finitely many maximal ideals. Hence $\text{Spec } A$ is a finite set and $\dim A \leq 1$. \square

5.0.10. The lemma above is used to prove the following lemma.

Lemma 5.3. *Let A be a local integral domain, $(0) \neq \mathfrak{m}$, and $\dim A = n \geq 1$. Let $f \in A \setminus \{0\}$ be a non-zero element with $f \in \mathfrak{m}$. Then there exists a prime ideal \mathfrak{p} such that $\dim A/\mathfrak{p} = 1$ and $f \notin \mathfrak{p}$.*

Proof. We will construct by induction on k a sequence of inclusion of prime ideals

$$(0) := \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k \subsetneq \mathfrak{m}$$

with the following conditions:

- (i) $f \notin \mathfrak{p}_k$;
- (ii) for every $1 \leq i \leq k$, the prime ideal \mathfrak{p}_i is of height one over \mathfrak{p}_{i-1} . In other words, there is no prime ideal \mathfrak{q} with $\mathfrak{p}_{i-1} \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_i$;
- (iii) either $\dim A/\mathfrak{p}_k = 1$ or $\dim A/\mathfrak{p}_k \leq (\dim A) - k$.

Suppose such a \mathfrak{p}_k is constructed, we see that the second condition in (iii) is always a consequence of (ii). This is because from the definition of height of a prime, one always has $k \leq \text{Ht}(\mathfrak{p}_k)$. By [dJP00, p 133, Remark 4.1.15], one always has

$$\text{Ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}_k) \leq \dim A.$$

Therefore,

$$k + \dim(A/\mathfrak{p}_k) \leq \text{Ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}_k) \leq \dim A,$$

and consequently $\dim(A/\mathfrak{p}_k) \leq (\dim A) - k$.

STEP 1: We let $A_0 := A/\mathfrak{p}_0 = A$. By hypothesis that $f \neq 0$ in A , therefore $(0) \in D_A(f)$, and so the open set $D_A(f)$ is non-empty. There are two cases according to whether $D_A(f)$ has exactly one or more than one elements:

(1) Suppose that $D_A(f) = \{(0)\}$, then set $\{(0)\}$ is Zariski open in $\text{Spec } A$. By Lemma 5.2 and remark in paragraph 5.0.6, one has $\dim A \leq 1$. By the hypothesis that $(0) \neq \mathfrak{m}$, hence $\dim A = 1$ and the proof is finished.

(2) Otherwise, the Zariski open set $D_A(f)$ contains another prime ideal \mathfrak{p}'_1 in A such that $\mathfrak{p}'_1 \neq (0)$. Moreover, since $\dim A$ is finite, $\text{Ht}(\mathfrak{p}')$ is finite, say h_1 . Consider a maximal chain of prime ideals

$$(0) = \mathfrak{p}_0 \subsetneq \mathfrak{q}_{1,1} \subsetneq \cdots \subsetneq \mathfrak{q}_{1,h_1} = \mathfrak{p}'_1$$

whose length is $\text{Ht}(\mathfrak{p}'_1)$. By maximality of the chain, $\text{Ht}(\mathfrak{q}_{1,1}) = 1$. Moreover, $f \notin \mathfrak{p}'_1$ implies that $f \notin \mathfrak{q}_{1,1}$. Therefore, we may let $\mathfrak{p}_1 := \mathfrak{q}_{1,1}$.

INDUCTIVE STEP: Once the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ have been constructed, the existence of \mathfrak{p}_{k+1} that satisfies the first two conditions (i) and (ii) will be constructed. The idea is to pass through the quotient $\pi_k : A \rightarrow A/\mathfrak{p}_k := A_k$ and repeat the steps as in **STEP 1**. This time, since $f \in \mathfrak{m}$ but $f \notin \mathfrak{p}_k$, hence $\mathfrak{p}_k \neq \mathfrak{m}$. So in A_k , the zero ideal (0) is not maximal. Also, the hypothesis that $f \notin \mathfrak{p}_k$ implies that the class f is not zero in A_k . Therefore, the open set $D_{A_k}(f)$ is not empty since it contains the zero ideal (0) . We study the open set $D_{A_k}(f)$ with the two following cases just as before:

(1) Either $D_{A_k}(f)$ is a singleton, meaning $D_{A_k}(f) = \{(0)\}$. Then $\{(0)\}$ is open in $\text{Spec } A_k$. By Lemma 5.2, one has $\dim A_k \leq 1$. But since $(0) \neq \mathfrak{m}$, $\dim A_k = 1$.

(2) Otherwise, in A_k we may find a prime ideal $\mathfrak{P}_{k+1} \subseteq A_k$ such that $\mathfrak{P}_k \in D_{A_k}(f)$ and $\mathfrak{P}_{k+1} \neq (0)$. Since $\dim A_k$ is finite, so is $\text{Ht}(\mathfrak{P}_{k+1})$, which is for example h_k . We let

$$(0) := \mathfrak{P}_0 \subsetneq \mathfrak{Q}_{k+1,1} \subsetneq \cdots \subsetneq \mathfrak{Q}_{k+1,h_k} := \mathfrak{P}_{k+1}$$

be a maximal chain of prime ideals in A_k corresponding whose length corresponds to the height of \mathfrak{P}_{k+1} . Therefore $\mathfrak{Q}_{k+1,1}$ is of height one by the maximality of the chain, and therefore we may let $\mathfrak{p}_{k+1} = \pi^{-1}(\mathfrak{Q}_{k+1,1})$. Moreover, since the class f does not belong to \mathfrak{P}_{k+1} , the element f does not belong to \mathfrak{p}_{k+1} . Hence \mathfrak{p}_{k+1} satisfies the two conditions and the inductive step is complete.

To conclude, since $\dim A = n$, the length of the chain k is at most $n - 1$. In this case one has

$$\dim A/\mathfrak{p}_{n-1} \leq 1.$$

But since $f \in \mathfrak{m}$ and $f \notin \mathfrak{p}_{n-1}$, so $\dim A/\mathfrak{p}_{n-1} = 1$, and the proof is complete. \square

5.0.11.

Lemma 5.4. Let $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{C}^n,0}$ be an ideal and $Y := V(\mathcal{I})$ be the variety defined by \mathcal{I} . Let f be a holomorphic function germ vanishing at the origin and $X = V(\mathcal{I} + (f))$ be the variety defined by $\mathcal{I} + (f)$. If the inclusion $X \subsetneq Y$ is strict, then there exists an irreducible curve $C \subseteq Y$ passing through the origin such that $f|_C \not\equiv 0$.

Proof. Almost half of the proof is done by the previous lemma. It suffices to observe that if the inclusion $X \subsetneq Y$ is strict, then there exists an irreducible component $W \subseteq Y$ passing through the origin on which $f|_W \not\equiv 0$. Once this is proved, then the condition that $f|_W \not\equiv 0$ implies that

$f \notin \mathcal{I}(W, 0)$. Since W is irreducible, the ideal $\mathcal{I}(W, 0)$ is prime. Moreover, the dimension of W is greater than 1, otherwise W will just be the origin but $f(0) = 0$, which contradicts that $f|_W \not\equiv 0$. Put differently, one has f a non-zero element in $\mathcal{O}_{W,0} = \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}(W, 0)$, with $f \in \mathfrak{m}$. Moreover, $\mathcal{O}_{W,0}$ is a local integral domain with $(0) \neq \mathfrak{m}$ since $\dim W \geq 1$. By previous lemma, there exists a prime ideal \mathfrak{P} in $\mathcal{O}_{W,0}$ such that the class f does not lie in \mathfrak{P} and $\dim \mathcal{O}_{W,0}/\mathfrak{P} = 1$. If we let $\pi : \mathcal{O}_{\mathbb{C}^n,0} \rightarrow \mathcal{O}_{W,0}$ be the usual ring homomorphism by quotient, one has the prime ideal $\mathfrak{p} := \pi^{-1}\mathfrak{P}$ such that $\dim \mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{p} = \dim \mathcal{O}_{W,0}/\mathfrak{P} = 1$, with $f \notin \mathfrak{p}$. Hence $C = V(\mathfrak{p})$ is an irreducible one-dimensional variety on which f does not totally vanish.

It remains to prove the observation. By Primary decomposition theorem, the ideal \mathcal{I} may be decomposed as an intersection

$$\mathcal{I} = \bigcap_{i=1}^M P_i$$

of primary ideals P_i , whose radical $\sqrt{P_i} := \mathfrak{p}_i$ is prime. Hence

$$Y = V(\mathcal{I}) = \bigcup_{i=1}^M V(P_i) = \bigcup_{i=1}^M V(\mathfrak{p}_i)$$

where the last equality follows from Nullstellensatz.

We claim that there exists i such that $f|_{V(\mathfrak{p}_i)} \not\equiv 0$. Otherwise, for every $1 \leq i \leq M$, one has $f \in \mathcal{I}(V(\mathfrak{p}_i), 0) = \mathfrak{p}_i$ by Nullstellensatz. Therefore, there exists a positive integer k such that for all $1 \leq i \leq M$, one has $f^k \in P_i$, and hence $f^k \in \mathcal{I}$. Therefore,

$$X = V(\mathcal{I} + (f)) = V(\mathcal{I}) \cap V(f) = V(\mathcal{I}) \cap V(f^k) = V(\mathcal{I} + (f^k)) = V(\mathcal{I}) = Y,$$

which contradicts our assumption that $X \subsetneq Y$. The proof is complete. \square

5.0.12. We have therefore arrived at one of the main results of this section.

Proposition 5.5. *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ with $f(0) = 0$. Then there exists $k \in \mathbb{N}$ such that*

$$f^k \in \left\langle \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right\rangle.$$

Proof. Let $X = V(\partial_{z_1} f, \dots, \partial_{z_n} f, f)$ and $Y = V(\partial_{z_1} f, \dots, \partial_{z_n} f)$. If $X \subsetneq Y$, by previous proposition, there exists an irreducible curve $C \subseteq Y$ such that $f|_C \not\equiv 0$. Let

$$\begin{aligned} h : (\mathbb{C}, 0) &\longrightarrow (C, 0) \\ \zeta &\longmapsto (h_1(\zeta), \dots, h_n(\zeta)) \end{aligned}$$

be a local normalisation of C . Hence

$$\frac{df \circ h(\zeta)}{d\zeta} = \sum_{k=1}^n \frac{\partial f}{\partial z_k} \circ h(\zeta) h'_k(\zeta).$$

Since $h(\mathbb{C}, 0) \subseteq C$, $(\partial_{z_k} f) \circ h(\zeta) \equiv 0$ for all $1 \leq k \leq n$. Therefore, $\frac{d}{d\zeta}(f \circ h) \equiv 0$, and hence $f \circ h$ is a constant on C . Moreover, since $f \circ h(0) = 0$, $f \circ h$ vanishes identically on C , which contradicts our hypothesis that $f|_C \not\equiv 0$. Therefore, $X = Y$ and by Nullstellensatz, there exists an integer k such that $f^k \in \mathcal{I}$ and the proof is complete. \square

5.1. Ideals Generated by Components of Gradients: Effective Aspects.

5.1.1. In fact, the exponent k in the previous proposition may be taken to be $k = n = \dim \mathbb{C}^n$. We summarise some of the details in [JP96, p 59].

5.1.2.

Definition 5.6. Let \mathcal{I} be an ideal in $\mathcal{O}_{\mathbb{C}^n,0}$. The *integral closure* of \mathcal{I} , denoted by $\bar{\mathcal{I}}$, is the set of germs $u \in \mathcal{O}_{\mathbb{C}^n,0}$ such that there exist $d \in \mathbb{N}_{\geq 1}$ and $\alpha_s \in \mathcal{I}^s$ for $1 \leq s \leq d$ with:

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0.$$

Definition 5.7. Let $\mathcal{I} = \langle F_1, \dots, F_N \rangle$ be an ideal of $\mathcal{O}_{\mathbb{C}^n,0}$ generated by N elements. Let $u \in \mathbb{R}_+$. The ideal $\bar{\mathcal{I}}^{(k)}$ is defined by

$$\bar{\mathcal{I}}^{(k)} = \{u \in \mathcal{O}_{\mathbb{C}^n,0} : |u| \leq C|F|^k\}$$

for some constants $C \geq 0$, and where $|F|^2 = |F_1|^2 + \cdots + |F_N|^2$.

5.1.3. By [JP96, p 60, Proposition 12.2], for every k, l positive real numbers, $\mathcal{I}^{(k)} \cdot \mathcal{I}^{(l)} \subseteq \mathcal{I}^{(k+l)}$. Moreover, $\bar{\mathcal{I}}^{(1)} = \bar{\mathcal{I}}$ which is the integral closure of the ideal \mathcal{I} ([JP96, p 61, Corollary 12.5]).

5.1.4. By the Briançon-Skoda theorem (1974), if $p = \min\{n-1, N-1\}$, then $\bar{\mathcal{I}}^{(k+p)} \subseteq \mathcal{I}^k$ for all $k \in \mathbb{N}$.

5.1.5. Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be a holomorphic function germ at the origin, and let \mathcal{I}_f and $J(f)$ denote the following ideals:

$$\begin{aligned} \mathcal{I}_f &:= \left\langle z_1 \frac{\partial f}{\partial z_1}, \dots, z_n \frac{\partial f}{\partial z_n} \right\rangle, \\ J(f) &:= \left\langle \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right\rangle. \end{aligned}$$

It is evident that $\mathcal{I}_f \subset J(f)$. Moreover, one has $f \in \bar{\mathcal{I}}_f = \mathcal{I}_f^{(1)}$ ([JP96, p 62, Corollary 12.6]). Therefore, by Paragraph 5.1.3, $f^{k+n-1} \in \mathcal{I}_f^{(k+n-1)}$. By Briançon-Skoda theorem, for all $k \in \mathbb{N}_{\geq 1}$

$$f^{k+n-1} \in \mathcal{I}_f^k.$$

By setting $k = 1$, $f^n \in J(f)$. This completes the proof of the following proposition:

Proposition 5.8. Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ with $f(0) = 0$. Then

$$f^n \in \left\langle \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right\rangle.$$

5.2. Application.

5.2.1. Let F_1, \dots, F_N be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n,0}$ such that intersection multiplicity of the ideal $\langle F_1, \dots, F_N \rangle$ is finite with data (p, q, s) . We will show that the ideal

$$\left\langle \frac{\partial F_i}{\partial z_j} : 1 \leq i \leq N, 1 \leq j \leq n \right\rangle$$

has an effectively bounded intersection multiplicity. More precisely,

Proposition 5.9. *For any $n \in \mathbb{N}_{\geq 1}$, let F_1, \dots, F_N be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n, 0}$ vanishing at the origin such that the ideal $\langle F_1, \dots, F_N \rangle$ has finite intersection multiplicity with data (p, q, s) . Then*

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} \Big/ \left\langle \frac{\partial F_i}{\partial z_j} : 1 \leq i \leq N, 1 \leq j \leq n \right\rangle \leq \binom{(n^2 + 2n)s + n - 1}{(n^2 + 2n)s - 1}.$$

Proof. By Proposition 5.8, for each $1 \leq i \leq N$,

$$F_i^n \in \left\langle \frac{\partial F_i}{\partial z_1}, \dots, \frac{\partial F_i}{\partial z_n} \right\rangle.$$

Evidently,

$$\langle F_1^n, \dots, F_N^n \rangle \subseteq \left\langle \frac{\partial F_i}{\partial z_j} : 1 \leq i \leq N, 1 \leq j \leq n \right\rangle.$$

As a result,

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} \Big/ \left\langle \frac{\partial F_i}{\partial z_j} : 1 \leq i \leq N, 1 \leq j \leq n \right\rangle \leq \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} \Big/ \langle F_1^n, \dots, F_N^n \rangle.$$

It suffices to estimate the term on the right. By the hypothesis that

$$|z|^p \lesssim \sum_{i=1}^N |F_i|,$$

Jensen's inequality yields

$$\begin{aligned} |z^{np}| &\lesssim \left(\sum_{i=1}^N |F_i| \right)^n \\ &= N^n \left(\sum_{i=1}^N \frac{1}{N} |F_i| \right)^n \\ &\leq N^n \sum_{i=1}^N \frac{1}{N} |F_i|^n \\ &= N^{n-1} \sum_{i=1}^N |F_i|^n \lesssim \sum_{i=1}^N |F_i^n|. \end{aligned}$$

By Theorem 4.7, the ideal $\langle F_1^n, \dots, F_N^n \rangle$ has finite intersection multiplicity with data (p', q', s') . By the definition of p' , one has $p' \leq np$. At the same time, by Proposition 4.11,

$$s' \leq \binom{n + q' + 1}{q' - 1}.$$

Also by Proposition 4.11, using $p' \leq np$, the q' in the inequality above has a bound

$$q' \leq (n + 2)p' \leq (n + 2)np \leq (n + 2)nq \leq (n + 2)ns.$$

Hence

$$s' \leq \binom{n + (n + 2)ns - 1}{(n + 2)ns - 1}$$

and the proof is complete. □

5.2.2. *Remark.* The estimate can be made more precise in $N = n$. In this case by repeated application of Proposition 4.17 ,

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \langle F_1^n, \dots, F_n^n \rangle = n^n \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \langle F_1, \dots, F_n \rangle = n^n s.$$

Therefore,

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \left\langle \frac{\partial F_i}{\partial z_j} : 1 \leq i, j \leq n \right\rangle \leq n^n s.$$

6. MULTIPLICITY OF AN IDEAL

6.0.1. Following [Chi89], we will present the notion of multiplicity of an ideal of holomorphic functions defining a pure dimensional variety. Let $F \in \mathcal{O}_{\mathbb{C}^n,0}$ be a holomorphic function germ. We may write F as an infinite sum of homogeneous polynomials

$$F = \sum_{k=m}^{\infty} F_k,$$

where each F_k is of degree k , with $F_m \neq 0$. The multiplicity of F at 0 is then equal to m .

6.0.2. Another way to characterise multiplicity is to look at the order of vanishing of F along generic lines. Indeed, let

$$\begin{aligned} \varphi : \mathbb{C} &\longrightarrow \mathbb{C}^n \\ \zeta &\longmapsto (c_1 \zeta, \dots, c_n \zeta) \end{aligned}$$

be a parametrisation of a line. Composing φ with F gives

$$F \circ \varphi = \sum_{k=m}^{\infty} F_k(c_1 \zeta, \dots, c_n \zeta) = \sum_{k=m}^{\infty} \zeta^k F_k(c_1, \dots, c_n).$$

Therefore, any vector (c_1, \dots, c_n) satisfying

$$F_m(c_1, \dots, c_n) \neq 0$$

will imply that $\text{ord}_0 F \circ \varphi = m = \text{mult}_0 F$. Since every line is a complete intersection of $n - 1$ hyperplanes H_1, \dots, H_{n-1} , the multiplicity of F is the intersection multiplicity of $V(F) = \{z \in \mathbb{C}^n : F(z) = 0\}$ with $n - 1$ generic hyperplanes. In other words, if $H_i = \{z \in \mathbb{C}^n : L_i = 0\}$ for some linear function L_i , then

$$\text{mult}_0 F = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \langle F, L_1, \dots, L_{n-1} \rangle.$$

6.0.3. More generally at 0, for $1 \leq q \leq n - 1$, let $\mathcal{I}_F = \langle F_1, \dots, F_q \rangle$ be the ideal generated by q holomorphic function germs and assume that it forms a *regular sequence*⁴. We would like to find a positive integer m analogous to the multiplicity of a function such that for $(n - q)$ generic hyperplanes,

$$m = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \langle F_1, \dots, F_q, L_1, \dots, L_{n-q} \rangle.$$

The important point about F_1, \dots, F_q being a regular sequence is that the variety $V(\mathcal{I}_F)$ defined by the ideal \mathcal{I}_F is of *pure dimension* $n - q$, due to the property of *Cohen-Macaulayness*. This allows us to apply the results in [Chi89, Chapter 2].

⁴Let R be a local ring. A sequence of non-units f_1, \dots, f_k is called a regular sequence if for all $1 \leq i \leq k$, the class f_i is not a zero divisor of $R / \langle f_1, \dots, f_{i-1} \rangle$

Definition 6.1 (Tangent Cones, [Chi89, p. 79]). Let E be an arbitrary set in \mathbb{R}^n . A vector $v \in \mathbb{R}^n$ is called *tangent* to E at a point of the closure $a \in \bar{E}$ if there exist a sequence of points $a_j \in E$ and positive numbers $t_j > 0$ such that $a_j \rightarrow a$ and

$$t_j(a_j - a) \rightarrow v \quad (j \rightarrow \infty).$$

The set of all such tangent vectors at a is denoted by $C(E, a)$, and is called the *tangent cone* to E at the point a .

6.0.4. The set $C(E, a)$ is a cone with vertex 0: if $v \in C(E, a)$, then the vectors tv lie in $C(E, a)$ for all $t > 0$. Geometrically, the cone is a set of limit positions of secants of E passing through a .

6.0.5. If V is a pure one-dimensional analytic set in \mathbb{C}^n , the tangent cone at any $a \in V$ is a finite union of complex lines ([Chi89, p. 80, Corollary]).

6.0.6. In general, if $0 \in V$ is a pure analytic subset of a domain D in \mathbb{C}^n , then $C(V, 0)$ is a pure p -dimensional algebraic set in \mathbb{C}^n (c.f. [Chi89, p. 83, Corollary]).

6.0.7. We recall that if a $n - p$ -dimensional variety V is defined by p holomorphic functions F_1, \dots, F_p which form a regular sequence, then V is a pure $(n - p)$ -dimensional analytic variety.

6.1. Multiplicities of Analytic Sets. We refer the readers to [Chi89, p. 120] for more details.

6.1.1. Let V be a pure p dimensional analytic set in \mathbb{C}^n , and let $a \in V$. Let L be an $(n - p)$ -dimensional complex subspace in \mathbb{C}^n , such that a is an isolated point of the set $V \cap (a + L)$. Then there is a domain $U \ni a$ in \mathbb{C}^n of the form $U = U' \times U'' \subseteq \mathbb{C}^{n-p} \times \mathbb{C}^p$ such that $V \cap U \cap (a + L) = \{a\}$, and the projection

$$\pi_L : V \cap U \rightarrow U' \subseteq L^\perp$$

along L is a ramified k -sheeted analytic cover. This number k is the multiplicity of the projection $\pi_L|_V$ at a , denoted by $\mu_a(\pi_L|_V)$.

6.1.2. For simplicity, suppose that $0 \in V$ and $a = 0$ in the previous paragraph, the multiplicity of intersection of V with L is $\mu_0(\pi_L|_V)$. See [Chi89, p 139, Corollary] and [Chi89, p 140, Proposition 1].

Definition 6.2 (Multiplicity of an Analytic Set at a point). Let V be a pure p -dimensional analytic set in \mathbb{C}^n and let $a \in V$. For every $(n - p)$ -dimensional plane L which contains the origin such that

$$V \cap (a + L) = \{a\},$$

the multiplicity of the projection $\mu_a(\pi_L|_V)$ is finite. The multiplicity of V at a is given by

$$\mu_a(V) := \min\{\mu_a(\pi_L|_V) : L \in G(n - p, n)\}.$$

Example 6.3. Suppose $V = \{F = 0\}$ is a principal analytic set in a neighbourhood of $0 \in \mathbb{C}^n$, and F is the minimum defining function for V . Write F

$$F = \sum_{k=\text{ord}_0 F}^{\infty} F_k$$

as a sum of homogeneous polynomials F_k of degree k . Then by [Chi89, p. 83, Proposition 1],

$$C(V, 0) = \{F_{\text{ord}_0 F} = 0\}.$$

For any complex *line* L containing 0 , by [Chi89, p. 121, Proposition 1],

$$\mu_0(\pi_L|_V) = \text{ord}_0 F|_L \geq \text{ord}_0 F,$$

with equality if and only if $L \cap C(V, 0) = \{0\}$. In other words, if

$$\zeta \mapsto (c_1 \zeta, \dots, c_n \zeta)$$

is a parametrisation of the line L , then the line has trivial intersection with $C(V, 0)$ if and only if

$$F_{\text{ord}_0 F}(c_1, \dots, c_n) \neq 0.$$

This agrees with our intuition in paragraph 6.0.2.

6.1.3. More generally,

Proposition 6.4. *Let V be a pure p -dimensional analytic set in a neighbourhood of $0 \in \mathbb{C}^n$, and let $L \in G(n - p, n)$. The equality $\mu_0(\pi_L|_V) = \mu_0$ holds if and only if the plane L is transversal to V at 0 . In other words,*

$$L \cap C(V, 0) = \{0\}.$$

Proof. See [Chi89, p. 122, Proposition 2]. □

6.1.4. Combining paragraphs 6.0.6, 6.0.7, 6.1.2, and Proposition 6.4, we obtain

Proposition 6.5. *Let F_1, \dots, F_p be holomorphic function germs at the origin so that $F_1(0) = \dots = F_p(0) = 0$. Suppose that the sequence F_1, \dots, F_p is regular so that the variety defined by the intersection $V := V(F_1, \dots, F_p)$ is a pure $(n - p)$ -dimensional analytic variety. Then there exists an integer $\mu_0(V)$ such that for a generic choice of p hyperplanes given by the zeros of $n - p$ linear functions L_1, \dots, L_{n-p} , one has*

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / \langle F_1, \dots, F_p, L_1, \dots, L_{n-p} \rangle = \mu_0(A).$$

6.2. Multiplicity of an Ideal – Case of a Curve.

6.2.1. In this section, we will discuss more in depth of Proposition 6.5 in the case where $p = n - 1$. In other words, the ideal $\mathcal{I}_F = \langle F_1, \dots, F_{n-1} \rangle$ forms a regular sequence in $\mathcal{O}_{\mathbb{C}^n, 0}$ so that the variety $V(F_1, \dots, F_{n-1})$ is a pure 1-dimensional analytic variety, which is a union

$$V(F_1, \dots, F_{n-1}) = \bigcup_{k=1}^M Z_k$$

of its irreducible components Z_k .

6.2.2. For $1 \leq k \leq M$, since each Z_k is an irreducible curve, there exists a parametrisation

$$\begin{aligned} n_k : (\mathbb{C}, 0) &\rightarrow (Z_k, 0) \\ \zeta &\mapsto (\zeta^{\mu_{k1}} a_{k1}(\zeta), \dots, \zeta^{\mu_{kn}} a_{kn}(\zeta)), \end{aligned}$$

where for all $1 \leq j \leq n$, $a_{kj}(0) \neq 0$ (c.f. [dJP00, p 164, Theorem 4.4.8]) and [dJP00, p 165, Theorem 4.4.10]).

Theorem 6.6. *There exist positive integers m_1, \dots, m_M such that for any holomorphic function f with*

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / \langle F_1, \dots, F_{n-1}, f \rangle < \infty,$$

the equality holds

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / \langle F_1, \dots, F_{n-1}, f \rangle = \sum_{k=1}^M m_k \operatorname{ord}_0 f \circ n_k.$$

Proof. See [D'A93, p 78, Theorem 3] for further discussion. □

6.2.3. We begin discussion with a small lemma.

Lemma 6.7. *Let Z_k be an irreducible 1-dimensional analytic variety and $n_k : (\mathbb{C}, 0) \rightarrow (Z_k, 0)$ be its normalisation. Let f be a holomorphic function germ vanishing at the origin such that $\operatorname{ord}_0 f \circ n_k$ is finite. Then the intersection $Z_k \cap \{f = 0\}$ is discrete, and hence the origin is an isolated point in the intersection.*

Proof. There is an equality of sets:

$$\{f = 0\} \cap Z_k = \{n_k(\zeta) : f \circ n_k(\zeta) = 0\}.$$

Now the set on the right is just simply $\{0\}$. This is because by hypothesis on the vanishing order of $f \circ n_k$,

$$f \circ n_k(\zeta) = \zeta^m g(\zeta),$$

where $g(0) \neq 0$ and $m = \operatorname{ord}_0 f \circ n_k < \infty$. Hence

$$f \circ n_k(\zeta) = 0 \implies \zeta = 0 \implies n_k(\zeta) = 0. \quad \square$$

Proposition 6.8. *Let f be the holomorphic function such that for each $1 \leq k \leq M$, the vanishing order $\operatorname{ord}_0 f \circ n_k$ is finite. Then the intersection multiplicity of the ideal $\langle F_1, \dots, F_{n-1}, f \rangle$ is finite.*

Proof. The previous lemma implies that

$$Z_k \cap \{f = 0\} = \{0\}.$$

Hence

$$V(F_1, \dots, F_{n-1}, f) = C \cap \{f = 0\} = \left(\bigcup_{k=1}^M Z_k \right) \cap \{f = 0\} = \bigcup_{k=1}^M (Z_k \cap \{f = 0\}) = \{0\}. \quad \square$$

6.2.4. We are now in a position to prove the following lemma.

Proposition 6.9. *Let F_1, \dots, F_{n-1} be a regular sequence such that $V(F_1, \dots, F_{n-1})$ is a pure 1-dimensional variety. For a generic choice of hyperplane defined by a linear function L ,*

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / \langle F_1, \dots, F_{n-1}, L \rangle = \sum_{k=1}^M m_k \min\{\mu_{k1}, \dots, \mu_{kn}\}.$$

Proof. First of all, L may be written as

$$L = \sum_{j=1}^n c_j z_j \quad (c_k \in \mathbb{C}).$$

Suppose that the intersection multiplicity of the ideal $\langle F_1, \dots, F_{n-1}, L \rangle$ is finite, by Theorem 6.6,

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / \langle F_1, \dots, F_{n-1}, L \rangle = \sum_{k=1}^M m_k \operatorname{ord}_0 L \circ n_k.$$

By Proposition 6.8, it suffices to choose an appropriate L such that $\operatorname{ord} L \circ n_k < \infty$. First, observe that

$$(6.10) \quad \begin{aligned} L \circ n_k &= \sum_{j=1}^n c_j \zeta^{\mu_{kj}} a_{kj}(\zeta) \\ &= \zeta^{\min\{\mu_{k1}, \dots, \mu_{kn}\}} \sum_{\{j: \mu_{kj} = \min\{\mu_{k1}, \dots, \mu_{kn}\}\}} c_j a_{kj}(\zeta) + O(\zeta^{\min\{\mu_{k1}, \dots, \mu_{kn}\} + 1}). \end{aligned}$$

If

$$(c_1, \dots, c_n) \in \mathbb{C}^n - \bigcup_{k=1}^M \left\{ (d_1, \dots, d_n) \in \mathbb{C}^n : \sum_{j=1}^n d_j a_{kj}(0) = 0 \right\},$$

then by equation (6.10),

$$\operatorname{ord}_0 L \circ n_k = \min\{\mu_{k1}, \dots, \mu_{kn}\} < \infty.$$

This completes the proof. \square

7. GENERIC SELECTION OF LINEAR COMBINATIONS FOR EFFECTIVE TERMINATION

The following proposition appears in [Siu10, p 1190].

Proposition 7.1. *Let $0 \leq q \leq n$, and f_1, \dots, f_q be holomorphic function germs on \mathbb{C}^n at the origin such that the common zero set $\{f_1 = \dots = f_q = 0\}$ is a pure $(n - q)$ -dimensional variety germ in \mathbb{C}^n at the origin. Let m be the multiplicity of the ideal $\langle f_1, \dots, f_q \rangle$ in the sense that for any $(n - q)$ generic homogeneous linear functions L_1, \dots, L_{n-q} ,*

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / \langle f_1, \dots, f_q, L_1, \dots, L_{n-q} \rangle = m.$$

Let $V(f_1, \dots, f_q, L_1, \dots, L_{n-q})$ be a pure 1 dimensional analytic variety and let

$$V(f_1, \dots, f_q, L_1, \dots, L_{n-q}) = \bigcup_{k=1}^M Z_k$$

be the irreducible decomposition of $V(f_1, \dots, f_q, L_1, \dots, L_{n-q})$. Let F_1, \dots, F_N be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n, 0}$ vanishing at the origin and $p \geq 1$ be an integer such that

$$|z|^p \lesssim \sum_{i=1}^N |F_i|.$$

Then there exist M hyperplanes, H_1, \dots, H_M , in \mathbb{C}^N such that for any

$$(c_1, \dots, c_N) \in \mathbb{C}^N - \bigcup_{i=1}^M H_i,$$

and for any generic $(n - q - 1)$ homogeneous linear functions L_1, \dots, L_{n-q-1} the following inequality holds

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / \left\langle f_1, \dots, f_q, \sum_{j=1}^N c_j F_j, L_1, \dots, L_{n-q-1} \right\rangle \leq mp.$$

Proof. As in the statement of the proof, let

$$V(f_1, \dots, f_q, L_1, \dots, L_{n-q-1}) = \bigcup_{k=1}^M Z_k$$

be the irreducible decomposition of the pure 1-dimensional analytic variety, and let

$$n_k : (\mathbb{C}, 0) \longrightarrow (Z_k, 0)$$

be normalisations of Z_k . By Theorem 6.6 and Proposition 6.8, there exist strictly positive integers m_1, \dots, m_M such that for any holomorphic function germs $g \in \mathcal{O}_{\mathbb{C}^n, 0}$ with $\text{ord}_0 g \circ n_k < \infty$ for all k ,

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / \langle f_1, \dots, f_q, g, L_1, \dots, L_{n-q-1} \rangle = \sum_{k=1}^M m_k \text{ord}_0 g \circ n_k.$$

It suffices to find suitable constants (c_1, \dots, c_N) such that the order of vanishing of the following function

$$g \circ n_k := \sum_{j=1}^N c_j F_j \circ n_k$$

is finite for all k . For each fixed $1 \leq k \leq M$, the map n_k may be explicitly written as

$$n_k : \zeta \longmapsto (\zeta^{\mu_{k,1}} a_{k,1}(\zeta), \dots, \zeta^{\mu_{k,n}} a_{k,n}(\zeta)),$$

where for each $1 \leq l \leq n$, $a_{k,l}(0) \neq 0$. Let

$$s_k := \min \{ \mu_{k,1}, \dots, \mu_{k,n} \} \quad (1 \leq k \leq M).$$

Pulling back the inequality

$$|z|^p \lesssim \sum_{j=1}^N |F_j|$$

by the normalisations give

$$\begin{aligned} |\zeta|^{s_k p} &\lesssim |(\zeta^{\mu_{k,1}} a_{k,1}(\zeta), \dots, \zeta^{\mu_{k,n}} a_{k,n}(\zeta))|^p \\ &\leq A \sum_{j=1}^N |F_j \circ n_k(\zeta)| \end{aligned} \quad (1 \leq k \leq M).$$

Consequently, not all $F_j \circ n_k$ vanish at the same time. For any $F_j \circ n_k \not\equiv 0$, the one-variable holomorphic function may be expanded into power series

$$F_j \circ n_k = \sum_{l=t_{j,k}}^{\infty} F_{j,k,l} \zeta^l \quad (1 \leq j \leq N, 1 \leq k \leq M),$$

where $t_{j,k} = \text{ord}_0 F_j \circ n_k$ and $F_{j,k,l} \in \mathbb{C}$. By convention, $t_{j,k} = \infty$ if $F_j \circ n_k \equiv 0$. For a fixed $1 \leq k \leq M$, let

$$t_k := \min \{ t_{j,k} : 1 \leq j \leq N, F_j \circ n_k \not\equiv 0 \} < \infty.$$

Hence

$$|\zeta|^{s_k p} \lesssim |\zeta|^{t_k} \quad (1 \leq k \leq M),$$

which implies that $t_k \leq s_k p$, since $|\zeta| \ll 1$. For any $(c_1, \dots, c_N) \in \mathbb{C}^N$,

$$\begin{aligned} \sum_{j=1}^N c_j F_j \circ n_k &= \sum_{j=1}^N \sum_{l \geq t_{j,k}}^{\infty} c_j F_{j,k,l} \zeta^l \\ &= \left(\sum_{\{j: t_{j,k}=t_k\}} c_j F_{j,k,t_k} \right) \zeta^{t_k} + O(\zeta^{t_k+1}) \end{aligned} \quad (1 \leq k \leq M).$$

Therefore, the order of vanishing of $\sum_{j=1}^N c_j F_j \circ n_k$ is exactly t_k if $\sum_{\{j: t_{j,k}=t_k\}} c_j F_{j,k,t_k} \neq 0$. If

$$(c_1, \dots, c_N) \in \mathbb{C}^N - \bigcup_{k=1}^M \left\{ (d_1, \dots, d_N) \in \mathbb{C}^N : \sum_{\{j: t_{j,k}=t_k\}} d_j F_{j,k,t_k} = 0 \right\}$$

which in the complement of the union of M hyperplanes, then for each $1 \leq k \leq M$,

$$\text{ord}_0 F \circ n_k = t_k \leq s_k p.$$

Consequently,

$$\begin{aligned} &\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} \left/ \left\langle f_1, \dots, f_q, \tilde{F}_1, \dots, \tilde{F}_\nu, \sum_{j=1}^N c_j F_j, L_1, \dots, L_{n-q-\nu-1} \right\rangle \right. \\ &= \sum_{k=1}^M m_k \text{ord}_0 \sum_{j=1}^N c_j F_j \circ n_k \\ &= \sum_{k=1}^M m_k t_k \leq \sum_{k=1}^M m_k s_k p \\ &= p \sum_{k=1}^M m_k s_k \\ &= p \sum_{k=1}^M m_k \min\{\mu_{k,1}, \dots, \mu_{k,n}\}. \end{aligned}$$

By Proposition 6.9, the number $\sum_{k=1}^M m_k \min\{\mu_{k,1}, \dots, \mu_{k,n}\}$ is the intersection multiplicity of the curve $V(f_1, \dots, f_q, L_1, \dots, L_{n-q-1})$ with a generic hyperplane defined by $\{L_{n-q} = 0\}$. By hypothesis,

$$\sum_{k=1}^M m_k \min\{\mu_{k,1}, \dots, \mu_{k,n}\} = m$$

and this completes the proof. □

7.0.1. *In dimension 2.* We will state the corollary of Proposition 7.1 in the case of dimension 2.

Corollary 7.2. *Let F_1, \dots, F_N be holomorphic functions in $\mathcal{O}_{\mathbb{C}^2,0}$ such that the ideal $\mathcal{I}_F = \langle F_1, \dots, F_N \rangle$ has finite intersection multiplicity with data (p, q, s) . Then there exist generic constants $(c_1, \dots, c_N) \in \mathbb{C}^N$ such that*

$$\text{mult}_0 \left(\sum_{j=1}^N c_j F_j \right) \leq q \leq 4p.$$

Moreover, let $V(\tilde{F}_1) = \cup_{k=1}^M Z_k$ be the irreducible decomposition of the variety. Then there exist M hyperplanes H_1, \dots, H_M in \mathbb{C}^N such that for all $(d_1, \dots, d_N) \in \mathbb{C}^N - \cup_{i=1}^M H_i$,

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \left\langle \sum_{j=1}^N c_j F_j, \sum_{j=1}^N d_j F_j \right\rangle \leq 4p^2 \leq 4s^2.$$

Proof. First, there exists $1 \leq i \leq N$ such that $\text{mult}_0 F_i \leq q$. Otherwise, if $\text{mult}_0 F_i \geq q+1$ for every $1 \leq i \leq N$, then

$$\mathfrak{m}^q \subseteq \langle F_1, \dots, F_N \rangle \subseteq \mathfrak{m}^{q+1},$$

which is a contradiction. So let (c_1, \dots, c_N) be constants so that

$$\text{mult}_0 \left(\sum_{j=1}^N c_j F_j \right) \leq q \leq 4p,$$

where the last inequality follows from Proposition 4.11. Then the existence of M hyperplanes in \mathbb{C}^N and constants (d_1, \dots, d_N) so that the conclusion holds follow directly from the previous proposition, and the proof is complete. \square

8. PROPER MAPS AND PROJECTIONS

8.0.1. In this section, let h_1, \dots, h_n be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n,0}$ vanishing at the origin with

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \langle h_1, \dots, h_n \rangle =: s < \infty.$$

Hence the $(n-1)$ -tuple (h_1, \dots, h_{n-1}) forms a regular sequence. By Proposition 6.9, and by a suitable linear change of coordinates, there exists a positive integer m such that

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / \langle h_1, \dots, h_{n-1}, z_n \rangle =: m$$

which is the multiplicity of the ideal $\langle h_1, \dots, h_{n-1} \rangle$.

8.0.2. The map

$$\begin{aligned} \varphi : (\mathbb{C}^n, 0) &\longrightarrow (\mathbb{C}^n, 0) \\ (z_1, \dots, z_n) &\longmapsto (h_1(z), \dots, h_{n-1}(z), z_n) =: (w_1, \dots, w_{n-1}, w_n) \end{aligned}$$

is proper and open with finite fibres. Let $H = \{h_n = 0\}$ be the hypersurface defined by the zeros of h_n . By Remmert's proper mapping theorem⁵, the image $\varphi(H)$ is also an analytic set. Since the map restricted to the hypersurface H :

$$\varphi|_H : H \longrightarrow \varphi(H)$$

⁵Remmert's proper mapping theorem may be stated as follows: if M and N are complex manifolds, $f : M \rightarrow N$ a holomorphic map and $V \subset M$ an analytic variety such that $f|_V$ is proper, then $f(V)$ is an analytic subvariety of N .

is surjective with finite fibres, by section 4, paragraph 5.0.8 (or [dJP00, p 129, Lemma 4.1.4]), one has $\dim H = \dim \varphi(H) = n - 1$.

8.0.3. Since $\varphi(H)$ is of dimension $n - 1$, it is a hypersurface locally defined at the origin by a certain holomorphic function $\tilde{h}_n(w_1, \dots, w_n)$, which will be shown to have the following properties:

(i) $\tilde{h}_n(0, \dots, 0, w_n) \not\equiv 0$ with certain order of vanishing $\lambda := \text{ord}_0(\tilde{h}_n(0, z_n))$. By the Weierstrass Preparation Theorem, \tilde{h}_n may be expressed as

$$\tilde{h}_n(w) = u(w) \left(w_n^\lambda + \sum_{j=0}^{\lambda-1} a_j(w_1, \dots, w_{n-1}) w_n^j \right),$$

for some unit $u(w)$, and $a_j(0) = 0$ for all $0 \leq j \leq \lambda - 1$.

(ii) $\lambda \leq s$.

8.0.4.

Lemma 8.1. *Let h_1, \dots, h_n be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n, 0}$ vanishing at the origin such that the intersection multiplicity of the ideal $\langle h_1, \dots, h_n \rangle$ is finite with data (p, q, s) . Let $H := \{h_n = 0\}$ be the hypersurface defined as the vanishing locus of h_n . Consider the map:*

$$\begin{aligned} \psi : H &\longrightarrow \mathbb{C}^{n-1} \\ z := (z_1, \dots, z_n) &\longmapsto (h_1(z), \dots, h_{n-1}(z)). \end{aligned}$$

Then there exists a open neighbourhood $U \subseteq \mathbb{C}^{n-1}$ of the origin $0 \in \mathbb{C}^{n-1}$ such that for every $\alpha := (\alpha_1, \dots, \alpha_{n-1}) \in U$, there are at most s distinct elements in $\psi^{-1}(\alpha)$.

Proof. We prove by contradiction. Suppose for every open neighbourhood $U \subseteq \mathbb{C}^{n-1}$ of the origin $0 \in \mathbb{C}^{n-1}$, there exists a point $\alpha \in U$ such that the number of distinct elements in $\psi^{-1}(\alpha)$ is at least $s + 1$.

By hypothesis, the map

$$\begin{aligned} \Psi : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ z &\longmapsto (h_1(z), \dots, h_n(z)) \end{aligned}$$

is a ramified s -sheeted analytic covering map. Hence, there exists a neighbourhood $V = V' \times V'' \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$ of the origin $0 \in \mathbb{C}^n$ such that for every $\beta := (\beta_1, \dots, \beta_n) \in V$, the number of distinct points in $\Psi^{-1}(\beta)$ is at most s .

But by our assumption, given V' a neighbourhood of the origin $0 \in \mathbb{C}^{n-1}$, there exists a point $\alpha := (\alpha_1, \dots, \alpha_{n-1}) \in V'$ such that there are at least $s + 1$ distinct points in $\psi^{-1}(\alpha)$. Since $(\alpha, 0) \in V$ and

$$\Psi^{-1}(\alpha, 0) = \psi^{-1}(\alpha),$$

there are at least $s + 1$ distinct points in $\Psi^{-1}(\alpha, 0)$, which is a contradiction. □

8.0.5. We will therefore answer the first claim in paragraph 3.

Proposition 8.2. *Let h_1, \dots, h_n be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n, 0}$ vanishing at the origin such that the multiplicity of the ideal $\langle h_1, \dots, h_n \rangle$ is $s \in \mathbb{N}_{\geq 1}$. Suppose that the holomorphic map*

$$\begin{aligned} \varphi : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\longmapsto (h_1(z), \dots, h_{n-1}(z), z_n) \end{aligned}$$

defines a ramified k -sheeted covering for some positive integer k . Let \tilde{h}_n be a holomorphic function such that $\varphi(\{h_n = 0\}) = \{\tilde{h}_n = 0\}$. Then $\tilde{h}_n(0, z_n) \not\equiv 0$.

Proof. Suppose on the contrary that $\tilde{h}_n(0, z_n) \equiv 0$. Consider the composition of maps

$$\begin{array}{ccccc} H & \xrightarrow{\varphi} & \varphi(H) & \xrightarrow{\text{proj}} & \mathbb{C}^{n-1} \\ (z_1, \dots, z_n) & \longmapsto & (h_1(z), \dots, h_{n-1}(z), z_n) & & \\ & & (w_1, \dots, w_n) & \longmapsto & (w_1, \dots, w_{n-1}). \end{array}$$

Here φ is the map in the statement of the proposition and proj is the projection onto the first $n - 1$ coordinates. Above $0 \in \mathbb{C}^{n-1}$, since $\tilde{h}_n(0, z_n) \equiv 0$,

$$\{(0, z_n) \in \mathbb{C}^n : z_n \in \mathbb{C}\} \subseteq \{\tilde{h}_n = 0\} = \varphi(H).$$

Moreover, since $\text{proj}(0, z_n) = 0 \in \mathbb{C}^{n-1}$,

$$\{(0, z_n) \in \mathbb{C}^n : z_n \in \mathbb{C}\} \subseteq \text{proj}^{-1}(0).$$

Therefore, $\text{proj}^{-1}(0)$ has infinitely many distinct fibre points. Consequently, $(\text{proj} \circ \varphi)^{-1}(0)$ has infinitely many distinct fibre points. But $\text{proj} \circ \varphi = \psi$ in the previous lemma, has finite distinct fibres, contradiction. \square

8.0.6. Next, we will show that $\text{ord}_0 \tilde{h}_n(0, z_n) \leq s$.

Lemma 8.3. *Let \tilde{h}_n be a holomorphic function germ in $\mathcal{O}_{\mathbb{C}^n, 0}$ with $\tilde{h}_n(0) = 0$, and $\tilde{h}_n(0, z_n) \not\equiv 0$ so that $\text{ord}_0 \tilde{h}_n(0, z_n) < \infty$. If the projection*

$$\begin{aligned} \pi : \{\tilde{h}_n = 0\} &\longrightarrow (\mathbb{C}^{n-1}, 0) \\ (\alpha_1, \dots, \alpha_n) &\longmapsto (\alpha_1, \dots, \alpha_{n-1}) \end{aligned}$$

is a finite surjective map with at most s distinct fibre points above each point in $(\mathbb{C}^{n-1}, 0)$, then $\text{ord}_0 \tilde{h}_n(0, z_n) \leq s$.

Proof. Suppose on the contrary that $\lambda := \text{ord}_0 \tilde{h}_n(0, z_n) \geq s + 1$. By the hypothesis that $\lambda < \infty$, Weierstrass Preparation Theorem implies the existence of a unit $u(z_1, \dots, z_n)$ and $\text{ord}_0 \tilde{h}_n(0, z_n)$ holomorphic functions $a_j(z_1, \dots, z_{n-1})$ vanishing at $(z_1, \dots, z_{n-1}) = (0, \dots, 0)$ such that

$$h(z_1, \dots, z_n) = u(z_1, \dots, z_n) \left(z_n^\lambda + \sum_{j=0}^{\lambda-1} a_j(z_1, \dots, z_{n-1}) z_n^j \right).$$

Therefore, above a generic point $(\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{C}^{n-1}$, the preimages $(\alpha_1, \dots, \alpha_{n-1}, z_n)$ of π which must satisfy the following polynomial equation

$$z_n^\lambda + \sum_{j=0}^{\lambda-1} a_j(\alpha_1, \dots, \alpha_{n-1}) z_n^j = 0$$

has $\lambda \geq s + 1$ distinct solutions in z_n . This contradicts the hypothesis in the statement of the lemma. \square

Proposition 8.4. *Let h_1, \dots, h_n be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^n, 0}$ vanishing at the origin such that*

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / \langle h_1, \dots, h_n \rangle = s < \infty.$$

Let $H = \{h_n = 0\}$. Suppose that the holomorphic map

$$\begin{aligned} \varphi : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\longmapsto (h_1(z), \dots, h_{n-1}(z), z_n) \end{aligned}$$

is proper, open so that there exists a holomorphic function $\tilde{h}_n(w_1, \dots, w_n)$ with $\varphi(H) = \{\tilde{h}_n = 0\}$. Then $\text{ord}_0 \tilde{h}_n(0, \dots, 0, w_n) \leq s$.

Proof. Consider the map

$$\begin{array}{ccccc} H & \xrightarrow{\varphi|_H} & \varphi(H) & \xrightarrow{\text{proj}} & \mathbb{C}^{n-1} \\ (z_1, \dots, z_n) & \longmapsto & (h_1(z), \dots, h_{n-1}(z), z_n) & & \\ & & (w_1, \dots, w_n) & \longmapsto & (w_1, \dots, w_{n-1}). \end{array}$$

By lemma 8.1, there exists a neighbourhood U of the origin $0 \in \mathbb{C}^{n-1}$ such that for all $\alpha \in U$, there are at most s distinct points in $\psi^{-1}(\alpha) = (\text{proj} \circ \varphi|_H)^{-1}(\alpha)$. Choose a generic point $\alpha \in \mathbb{C}^{n-1}$ as in the lemma 8.3. Therefore, above α , there are $\text{ord}_0 \tilde{h}_n(0, w_n)$ distinct fibre points in $\text{proj}^{-1}(\alpha)$, and hence

$$\begin{aligned} &\text{ord}_0 \tilde{h}_n(0, w_n) \\ &= \text{number of distinct points in } \text{proj}^{-1}(\alpha) \\ &\leq \text{number of distinct points in } (\text{proj} \circ \varphi|_H)^{-1}(\alpha) \leq s. \quad \square \end{aligned}$$

9. CALCULATION OF EXPLICIT ε IN DIMENSION 2 (PRELIMINARIES)

9.0.1. In this section we will use some of the results in the earlier sections to establish some preliminary results for the calculation of explicit ε in the case of dimension 2.

9.0.2. Let F_1, \dots, F_N be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^2, 0}$ vanishing at the origin such that the ideal they generate $\langle F_1, \dots, F_N \rangle$ has finite intersection multiplicity with data (p, q, s) .

9.1. Ideal Generated by Gradient and Generic Selection in Dimension 2.

9.1.1. In \mathbb{C}^2 , Proposition 5.9 implies that

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2, 0} / \left\langle \frac{\partial F_i}{\partial z_j} : 1 \leq i \leq N, 1 \leq j \leq 2 \right\rangle \leq \binom{8s+1}{8s-1}.$$

Moreover, if $N = 2$, there is a better upper bound

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2, 0} / \left\langle \frac{\partial F_i}{\partial z_j} : 1 \leq i, j \leq 2 \right\rangle \leq 4s.$$

9.1.2. Let \tilde{h}_2 be any holomorphic function germ in $\mathcal{O}_{\mathbb{C}^2,0}$ with multiplicity \tilde{m}_2 . Let

$$V(\tilde{h}_2) = \bigcup_{k=1}^{\tilde{r}} Z_k$$

be the irreducible decomposition of the pure 1-dimensional analytic variety. By Proposition 7.1, there exist \tilde{r} hyperplanes $H_1, \dots, H_{\tilde{r}}$ in \mathbb{C}^{2N} such that for all

$$(\lambda_1, \dots, \lambda_N, \theta_1, \dots, \theta_N) \in \mathbb{C}^{2N} - \bigcup_{i=1}^{\tilde{r}} H_i,$$

there is an effective upper bound on the intersection multiplicity

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} \Big/ \left\langle \tilde{h}_2, \sum_{j=1}^N \lambda_j \frac{\partial F_j}{\partial z_1} + \theta_j \frac{\partial F_j}{\partial z_2} \right\rangle \leq \tilde{m}_2 \binom{8s+1}{8s-1}.$$

9.1.3.

Lemma 9.1. *Let \tilde{h}_2 be any holomorphic function germ in $\mathcal{O}_{\mathbb{C}^2,0}$ that vanishes at the origin, whose multiplicity is \tilde{m}_2 . Suppose that the vanishing locus $\{\tilde{h}_2 = 0\}$ is a union of \tilde{r} irreducible components (not counting multiplicity). Then there exist $2\tilde{r}$ hyperplanes $H_1, \dots, H_{2\tilde{r}}$ in \mathbb{C}^N so that whenever*

$$(c_1, \dots, c_N) \in \mathbb{C}^N - \bigcup_{k=1}^{2\tilde{r}} H_k$$

there are \tilde{r} hyperplanes $\tilde{H}_1, \dots, \tilde{H}_{\tilde{r}}$ in \mathbb{C}^2 such that if

$$(\alpha, \gamma) \in \mathbb{C}^2 - \bigcup_{k=1}^{\tilde{r}} \tilde{H}_k,$$

then it holds that

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} \Big/ \left\langle \tilde{h}_2, \sum_{j=1}^N c_j \alpha \frac{\partial F_j}{\partial z_1} + c_j \gamma \frac{\partial F_j}{\partial z_2} \right\rangle \leq \tilde{m}_2 \binom{8s+1}{8s-1}.$$

Proof. By paragraph 9.1.1, the ideal

$$\left\langle \frac{\partial F_i}{\partial z_j} : 1 \leq i \leq N, 1 \leq j \leq 2 \right\rangle$$

has finite intersection multiplicity with data (p', q', s') .

By Proposition 7.1, there exist \tilde{r} hyperplanes in \mathbb{C}^{2N} of the form

$$H'_l = \left\{ (v_1, \dots, v_N, w_1, \dots, w_N) \in \mathbb{C}^{2N} : \sum_{k=1}^N \sigma_{lk} v_k + \mu_{lk} w_k = 0 \right\} \quad (1 \leq l \leq \tilde{r}),$$

such that if $(v_1, \dots, v_N, w_1, \dots, w_N) \in \mathbb{C}^{2N} - \bigcup_{l=1}^{\tilde{r}} H'_l$, then

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} \Big/ \left\langle \tilde{h}_2, \sum_{k=1}^{2N} v_k \frac{\partial F_k}{\partial z_1} + w_k \frac{\partial F_k}{\partial z_2} \right\rangle \leq \tilde{m}_2 \binom{8s+1}{8s-1}.$$

To conclude the proof, it suffices to choose $(c_1\alpha, \dots, c_N\alpha, c_1\gamma, \dots, c_N\gamma) \in \mathbb{C}^{2N} - \cup_{l=1}^{\tilde{r}} H'_l$, or equivalently for every $1 \leq l \leq \tilde{r}$,

$$(9.2) \quad \sum_{k=1}^N \sigma_{lk} c_k \alpha + \mu_{lk} c_k \gamma \neq 0.$$

To this aim, write

$$(9.3) \quad \sum_{k=1}^N \sigma_{lk} c_k \alpha + \mu_{lk} c_k \gamma = \left(\sum_{k=1}^N \sigma_{lk} c_k \right) \alpha + \left(\sum_{k=1}^N \mu_{lk} c_k \right) \gamma.$$

If

$$\begin{aligned} (c_1, \dots, c_N) \in \mathbb{C}^N & - \bigcup_{l=1}^{\tilde{r}} \left\{ (d_1, \dots, d_N) \in \mathbb{C}^N : \sum_{k=1}^N \sigma_{lk} d_k = 0 \right\} \\ & - \bigcup_{l=1}^{\tilde{r}} \left\{ (d_1, \dots, d_N) \in \mathbb{C}^N : \sum_{k=1}^N \mu_{lk} d_k = 0 \right\}, \end{aligned}$$

which is in a complement of $2\tilde{r}$ hyperplanes, the coefficients of α and γ in the equation (9.3) do not vanish. Once (c_1, \dots, c_N) is chosen, if

$$(\alpha, \gamma) \in \mathbb{C}^2 - \bigcup_{l=1}^{\tilde{r}} \left\{ \left(\sum_{k=1}^N \sigma_{lk} c_k \right) \alpha + \left(\sum_{k=1}^N \mu_{lk} c_k \right) \gamma = 0 \right\},$$

which lies in the complement of \tilde{r} hyperplanes in \mathbb{C}^2 , then equation (9.2) holds. Hence the proof is complete. \square

9.1.4.

Proposition 9.4. *Let $(z_1, z_2) \in \mathbb{C}^2$ be holomorphic coordinates in \mathbb{C}^2 . Let \tilde{h}_2 be a holomorphic function germ in $\mathcal{O}_{\mathbb{C}^2, 0}$ vanishing at the origin with multiplicity \tilde{m}_2 , and suppose that its vanishing locus $\{\tilde{h}_2 = 0\}$ has \tilde{r} irreducible components (not counting multiplicity).*

Let F_1, \dots, F_N be holomorphic function germs which generate an ideal $\langle F_1, \dots, F_N \rangle$ having finite intersection multiplicity with data (p, q, s) .

Let

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

be an invertible linear change of coordinates. Then there are $3\tilde{r}$ hyperplanes $H_1, \dots, H_{3\tilde{r}}$ in \mathbb{C}^N such that for each $(c_1, \dots, c_N) \in \mathbb{C}^N - \cup_{k=1}^{3\tilde{r}} H_k$, there exist \tilde{r} hyperplanes $\tilde{H}_1, \dots, \tilde{H}_{\tilde{r}}$ and a hypersurface defined by a homogeneous polynomial $\{P = 0\}$ such that whenever

$$(\alpha, \gamma) \in \mathbb{C}^2 - \bigcup_{k=1}^{\tilde{r}} \tilde{H}_k - \{P = 0\},$$

the linear combination

$$h_1(z_1, z_2) = \sum_{j=1}^N c_j F_j(z_1, z_2)$$

will satisfy the following conditions:

(i) the intersection multiplicity of the ideal $\langle h_1, \tilde{h}_2 \rangle$ has an effective bound:

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \langle h_1, \tilde{h}_2 \rangle \leq \tilde{m}_2 p \leq \tilde{m}_2 s;$$

(ii) in the new coordinates (w_1, w_2) ,

$$\begin{aligned} & \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \left\langle \tilde{h}_2(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), \frac{\partial h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)}{\partial w_1} \right\rangle \\ & \leq \tilde{m}_2 \binom{8s+1}{8s-1}; \end{aligned}$$

(iii) the holomorphic map induced from the change of coordinates

$$\begin{aligned} \varphi : \mathbb{C}^2 & \longrightarrow \mathbb{C}^2 \\ (w_1, w_2) & \longmapsto (h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), w_2) \end{aligned}$$

is a covering map with finite fibres.

Proof. (i) By Proposition 7.1, there exist \tilde{r} hyperplanes $H_1, \dots, H_{\tilde{r}}$ in \mathbb{C}^N so that for all

$$(c_1, \dots, c_N) \in \mathbb{C}^N - \bigcup_{k=1}^{\tilde{r}} H_k,$$

one has (in variables (z_1, z_2))

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \left\langle \tilde{h}_2, \sum_{j=1}^N c_j F_j \right\rangle \leq \tilde{m}_2 p \leq \tilde{m}_2 s.$$

This satisfies the first condition, which remains unchanged even after a linear change of coordinates $(z_1, z_2) \leftrightarrow (w_1, w_2)$.

(ii) After a change of variables,

$$\begin{aligned} \frac{\partial h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)}{\partial w_1} &= \frac{\partial h_1(z_1, z_2)}{\partial z_1} \frac{\partial z_1}{\partial w_1} + \frac{\partial h_1(z_1, z_2)}{\partial z_2} \frac{\partial z_2}{\partial w_1} \\ &= \alpha \frac{\partial h_1(z_1, z_2)}{\partial z_1} + \gamma \frac{\partial h_1(z_1, z_2)}{\partial z_2} \\ &= \sum_{j=1}^N c_j \alpha \frac{\partial F_j(z_1, z_2)}{\partial z_1} + c_j \gamma \frac{\partial F_j(z_1, z_2)}{\partial z_2}. \end{aligned}$$

By Lemma 9.1, there exist $2\tilde{r}$ hyperplanes $H_{\tilde{r}+1}, \dots, H_{3\tilde{r}}$ in \mathbb{C}^N such that whenever

$$(c_1, \dots, c_N) \in \mathbb{C}^N - \bigcup_{k=\tilde{r}+1}^{3\tilde{r}} H_k,$$

there are \tilde{r} hyperplanes $\tilde{H}_1, \dots, \tilde{H}_{\tilde{r}}$ in \mathbb{C}^2 so that if

$$(\alpha, \gamma) \in \mathbb{C}^2 - \bigcup_{k=1}^{\tilde{r}} \tilde{H}_k,$$

then

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} \left/ \left\langle \tilde{h}_2(z_1, z_2), \sum_{j=1}^N c_j \alpha \frac{\partial F_j(z_1, z_2)}{\partial z_1} + c_j \gamma \frac{\partial F_j(z_1, z_2)}{\partial z_2} \right\rangle \right. \leq \tilde{m}_2 \binom{8s+1}{8s-1},$$

or in other words,

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} \left/ \left\langle \tilde{h}_2(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), \frac{\partial h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)}{\partial w_1} \right\rangle \right. \leq \tilde{m}_2 \binom{8s+1}{8s-1},$$

and hence the second condition is attained.

(iii) For the last condition, in order for φ to be a covering map with finite fibres, it suffices to find $(\alpha, \gamma) \in \mathbb{C}^2$ so that the holomorphic function of one variable

$$h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)|_{\{w_2=0\}} = h_1(\alpha w_1, \gamma w_1)$$

has a finite order of vanishing at $w_1 = 0$. To this effect, $h_1(z_1, z_2)$ may be written as an infinite sum

$$h_1(z_1, z_2) = P_{m_1} + \sum_{k \geq m_1+1} P_k$$

of homogeneous polynomials P_k of degree k , with $m_1 = \text{mult}_0 h_1$. Hence,

$$\begin{aligned} h_1(\alpha w_1, \gamma w_1) &= P_{m_1}(\alpha w_1, \gamma w_1) + O(w_1^{m_1+1}) \\ &= w_1^{m_1} P_m(\alpha, \gamma) + O(w_1^{m_1+1}). \end{aligned}$$

If $(\alpha, \gamma) \in \mathbb{C}^2 - \{P_{m_1} = 0\}$, then $h_1(\alpha w_1, \gamma w_1) \not\equiv 0$ and hence φ defines a ramified m_1 -sheeted analytic covering.

In summary, there exist $3\tilde{r}$ hyperplanes $H_1, \dots, H_{3\tilde{r}}$ in \mathbb{C}^N so that for every

$$(c_1, \dots, c_N) \in \mathbb{C}^N - \bigcup_{k=1}^{3\tilde{r}} H_k,$$

there are \tilde{r} hyperplanes $\tilde{H}_1, \dots, \tilde{H}_{\tilde{r}}$ and a hypersurface $\{P_{m_1} = 0\}$ in \mathbb{C}^2 such that whenever

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

is an invertible linear change of coordinate satisfying

$$(\alpha, \gamma) \in \mathbb{C}^2 - \bigcup_{k=1}^{\tilde{r}} \tilde{H}_k - \{P_{m_1} = 0\},$$

the three conditions **(i)**, **(ii)**, **(iii)** are satisfied. □

10. EXPLICIT CALCULATION OF ε IN DIMENSION 2

10.0.1. As before, we work in \mathbb{C}^2 . Let F_1, \dots, F_N be holomorphic function germs in $\mathcal{O}_{\mathbb{C}^2,0}$ vanishing at the origin whose ideal $\mathcal{I}_F = \langle F_1, \dots, F_N \rangle$ has finite intersection multiplicity with data (p, q, s) .

10.0.2. By [Siu10, p 1182], one has for all $\phi \in \mathcal{D}_{0,1}(\Omega)$ with compact support that

$$\| |dF_j \cdot \phi| \|_{\frac{1}{4}}^2 \lesssim Q(\phi, \phi) \quad (1 \leq j \leq N). \quad \text{36}$$

10.0.3. For any two vectors $(\lambda_1, \dots, \lambda_N), (\mu_1, \dots, \mu_N)$ in \mathbb{C}^N , if

$$A = \sum_{i=1}^N \lambda_i F_i \quad \text{and} \quad B = \sum_{i=1}^N \mu_i F_i,$$

then by 3.2(iii)

$$\| \text{Jac}(A, B) \phi \|_{\frac{1}{4}}^2 \lesssim Q(\phi, \phi).$$

10.0.4. By Corollary 7.2, there exist vectors $(\lambda_1, \dots, \lambda_N)$ and (μ_1, \dots, μ_N) such that

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2, 0} / \langle A, B \rangle \leq 4s^2.$$

By Corollary 4.16,

$$\text{mult}_0 \text{Jac}(A, B) \leq 4s^2 - 1.$$

10.0.5. Write

$$(10.1) \quad \text{Jac}(A, B) = f_1^{\alpha_1} \cdots f_{\tilde{r}}^{\alpha_{\tilde{r}}}$$

as a product of prime elements, and let $\alpha := \max\{\alpha_1, \dots, \alpha_{\tilde{r}}\}$. The holomorphic function

$$\tilde{h}_2 := f_1 \cdots f_{\tilde{r}}$$

is also a subelliptic multiplier since

$$\tilde{h}_2^\alpha = f_1^\alpha \cdots f_{\tilde{r}}^\alpha = f_1^{\alpha - \alpha_1} \cdots f_{\tilde{r}}^{\alpha - \alpha_{\tilde{r}}} \text{Jac}(A, B)$$

is a multiple of a subelliptic multiplier. Consequently, by radical property of subelliptic multipliers Proposition 3.2(i),

$$\| \tilde{h}_2 \phi \|_{\frac{1}{4\alpha}}^2 \lesssim Q(\phi, \phi).$$

Moreover, by equation (10.1),

$$\text{mult}_0 \text{Jac}(A, B) = \sum_{i=1}^{\tilde{r}} \alpha_i \text{mult}_0 (f_i) \geq \alpha.$$

Hence

$$\frac{1}{4\alpha} \geq \frac{1}{4 \text{mult}_0 \text{Jac}(A, B)} \geq \frac{1}{4(4s^2 - 1)},$$

and

$$\| \tilde{h}_2 \phi \|_{\frac{1}{4(4s^2 - 1)}}^2 \lesssim \| \tilde{h}_2 \phi \|_{\frac{1}{4\alpha}}^2 \lesssim Q(\phi, \phi).$$

10.0.6. As a remark,

$$\text{mult}_0 \tilde{h}_2 = \sum_{i=1}^{\tilde{r}} \text{mult}_0 f_i \leq \sum_{i=1}^{\tilde{r}} \alpha_i \text{mult}_0 f_i = \text{mult}_0 \text{Jac}(A, B) \leq 4s^2 - 1.$$

10.0.7. By Proposition 9.4, there exists $(c_1, \dots, c_N) \in \mathbb{C}^N$ and a linear change of coordinate $(z_1, z_2) \mapsto (w_1, w_2)$ via

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

such that if $h_1 = \sum_{k=1}^N c_k F_k$, one has

(i)

$$\begin{aligned} & \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \langle h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), \tilde{h}_2(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2) \rangle \\ & \leq (\text{mult}_0 \tilde{h}_2) s \leq (4s^2 - 1)s; \end{aligned}$$

(ii)

$$\begin{aligned} & \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \left\langle h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), \frac{\partial h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)}{\partial w_1} \right\rangle \\ & \leq (\text{mult}_0 \tilde{h}_2) \binom{8s+1}{8s-1}; \end{aligned}$$

(iii) if we let $(\mathbb{C}^2, (w_1, w_2))$ [resp. $(\mathbb{C}^2, (x, y))$] denote \mathbb{C}^2 with coordinate system (w_1, w_2) [resp. (x, y)], the holomorphic map

$$\begin{aligned} \varphi : (\mathbb{C}^2, (w_1, w_2)) & \longrightarrow (\mathbb{C}^2, (x, y)) \\ (w_1, w_2) & \longmapsto (h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), w_2) \end{aligned}$$

defines a ramified $\text{ord}_{w_1=0} \tilde{h}_1(\alpha w_1, \beta w_1)$ -cover over \mathbb{C}^2 , which is therefore open and proper with finite fibres.

10.0.8. Since h_1 vanishes at the origin, by Lemma 4.12,

$$\begin{aligned} & h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)^{(4s^2-1)\binom{8s+1}{8s-1}} \\ & \in \left\langle \frac{\partial h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)}{\partial z_1}, \tilde{h}_2(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2) \right\rangle. \end{aligned}$$

10.0.9. Let $C_2 := \{\tilde{h}_2(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2) = 0\}$ be the reduced curve. Since φ is a proper map, by Remmert's proper mapping theorem, the image $\tilde{C}_2 := \varphi(C_2)$ is an analytic set of dimension 1. There exists an analytic function h_2 on \mathbb{C}^2 such that $\varphi(C_2) = \{h_2(x, y) = 0\}$. By Proposition 8.4, $\lambda := \text{ord}_0 h_2(0, y) \leq (4s^2 - 1)s$. Hence, by Weierstrass' preparation theorem, there exist a unit $u(x, y)$, and holomorphic functions $a_1(x), \dots, a_{(4s^2-1)s-1}(x)$ that vanish at $x = 0$ such that $h_2(x, y)$ may be expressed as a Weierstrass polynomial

$$h_2(x, y) = u(x, y) \left(y^\lambda + \sum_{k=0}^{\lambda-1} a_k(x) y^k \right) \quad \lambda \leq (4s^2 - 1)s.$$

10.0.10. The holomorphic function $h_2(h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), w_2)$ is also a subelliptic multiplier. More precisely, $h_2(h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), w_2)$ is a multiple of $\tilde{h}_2(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)$ which is a subelliptic multiplier by paragraph 10.0.5. This follows from the fact (which will be explained below) that $V(\tilde{h}_2) \subseteq V(h_2(h_1, w_2))$ and hence by the Nullstellensatz,

$$\langle h_2(h_1, w_2) \rangle \subseteq \sqrt{\langle h_2(h_1, w_2) \rangle} \subseteq \sqrt{\langle \tilde{h}_2 \rangle} = \langle \tilde{h}_2 \rangle,$$

where the equality follows from the fact that \tilde{h}_2 is reduced.

Now to show that $V(\tilde{h}_2(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)) \subseteq V(h_2(h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), w_2))$, if $(\sigma, \mu) \in \mathbb{C}^2$ satisfies $\tilde{h}_2(\alpha\sigma + \beta\mu, \gamma\sigma + \delta\mu) = 0$, then

$$\varphi(\sigma, \mu) = (h_1(\alpha\sigma + \beta\mu, \gamma\sigma + \delta\mu), \mu) \in \{h_2(x, y) = 0\}.$$

Hence

$$0 = h_2(\varphi(\sigma, \mu)) = h_2(h_1(\alpha\sigma + \beta\mu, \gamma\sigma + \delta\mu), \mu),$$

from which we have proved the set inclusion. Consequently,

$$\|h_2(h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), w_2)\phi\|_{\frac{1}{4(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

Moreover, since $u(h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2), w_2)$ is a unit,

$$\left\| \left(w_2^\lambda + \sum_{j=1}^{\lambda-1} a_j(h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)) w_2^j \right) \phi \right\|_{\frac{1}{4(4s^2-1)}}^2 \lesssim Q(\phi, \phi),$$

where $\lambda \leq (4s^2 - 1)s$.

10.0.11. To declutter notations, we will set

$$\begin{aligned} h_1(w_1, w_2) &:= h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2) \\ \frac{\partial h_1(w_1, w_2)}{\partial w_1} &:= \frac{\partial h_1(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)}{\partial w_1} \\ \tilde{h}_2(w_1, w_2) &:= \tilde{h}_2(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2) \\ \eta &:= (4s^2 - 1) \binom{8s+1}{8s-1} \\ \lambda &:= \text{ord}_0 h_2(0, y) \leq (4s^2 - 1)s. \end{aligned}$$

By Paragraph 10.0.8, since

$$h_1^\eta \in \left\langle \frac{\partial h_1}{\partial w_1}, \tilde{h}_2 \right\rangle,$$

there is an estimate

$$|h_1^\eta| \lesssim \left| \frac{\partial h_1}{\partial w_1} \right| + |\tilde{h}_2|.$$

10.1. Siu's method: Starting Point.

10.1.1. Since h_1 is a pre-multiplier and

$$dh_1 \wedge dh_2 = \frac{\partial h_1}{\partial w_1} \left(\lambda w_2^{\lambda-1} + \sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1} \right) dw_1 \wedge dw_2,$$

the holomorphic function

$$\frac{\partial h_1}{\partial w_1} \left(\lambda w_2^{\lambda-1} + \sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1} \right)$$

is also a subelliptic multiplier and we will estimate its regularity property. Since

$$\left\| \left(w_2^\lambda + \sum_{j=0}^{\lambda-1} a_j(h_1) w_2^j \right) \phi \right\|_{\frac{1}{4(4s^2-1)}}^2 \lesssim Q(\phi, \phi),$$

using Proposition 3.2(ii),

$$\left\| d \left(w_2^\lambda + \sum_{j=0}^{\lambda-1} a_j(h_1) w_2^j \right) \cdot \phi \right\|_{\frac{1}{8(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

Also,

$$\| |dh_1 \cdot \phi| \|_{\frac{1}{8(4s^2-1)}}^2 \lesssim \| |dh_1 \cdot \phi| \|_{\frac{1}{4}}^2 \lesssim Q(\phi, \phi),$$

where the last inequality comes from Paragraph 10.0.2 and the fact that h_1 is a linear combination of the F_i . By Proposition 3.2(iii), the regularity of the subelliptic multiplier is obtained below

$$\left\| \frac{\partial h_1}{\partial w_1} \left(\lambda w_2^{\lambda-1} + \sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1} \right) \phi \right\|_{\frac{1}{8(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

10.1.2. Since \tilde{h}_2 is also a subelliptic multiplier, so is

$$\tilde{h}_2 \left(\lambda w_2^{\lambda-1} + \sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1} \right).$$

Hence by paragraph 10.0.5,

$$\left\| \tilde{h}_2 \left(\lambda w_2^{\lambda-1} + \sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1} \right) \phi \right\|_{\frac{1}{4(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

10.1.3. By the previous two paragraphs and the inequality in 10.0.11,

$$|h_1^\eta| \lesssim \left| \frac{\partial h_1}{\partial z_1} \right| + |\tilde{h}_2|,$$

there is an estimate

$$\begin{aligned}
& \left\| \left\| h_1^\eta \left(\lambda w_2^{\lambda-1} + \sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1} \right) \phi \right\| \right\|_{\frac{1}{8(4s^2-1)}}^2 \\
& \lesssim \left\| \left\| \frac{\partial h_1}{\partial w_1} \left(\lambda w_2^{\lambda-1} + \sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1} \right) \phi \right\| \right\|_{\frac{1}{8(4s^2-1)}}^2 + \left\| \left\| \tilde{h}_2 \left(\lambda w_2^{\lambda-1} + \sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1} \right) \phi \right\| \right\|_{\frac{1}{8(4s^2-1)}}^2 \\
& \lesssim Q(\phi, \phi).
\end{aligned}$$

10.2. Siu's method: Inductive Step.

10.2.1. Let $h_2^{(0)} := h_2$ and

$$h_2^{(1)} := h_1^\eta \left(\lambda w_2^{\lambda-1} + \sum_{j=1}^{\lambda-1} j a_j(h_1) w_2^{j-1} \right).$$

For $1 \leq \nu \leq \lambda$, define

$$h_2^{(\nu)} := h_1^{\nu\eta} \left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu} \right),$$

which will be shown that it is also a subelliptic multiplier and

$$\| \| h_2^{(\nu)} \phi \| \|_{\frac{1}{2^{\nu \cdot 4}(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

10.2.2. We will first calculate something analogous to the first paragraph of the previous subsection. Suppose that the induction statement is true for $\nu - 1$, meaning that

$$\| \| h_2^{(\nu-1)} \phi \| \|_{\frac{1}{2^{\nu-1 \cdot 4}(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

Then

$$\| \| dh_2^{(\nu-1)} \phi \| \|_{\frac{1}{2^{\nu \cdot 4}(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

Moreover,

$$\| \| dh_1 \cdot \phi \| \|_{\frac{1}{4}}^2 \lesssim Q(\phi, \phi).$$

Therefore,

$$dh_1 \wedge dh_2^{(\nu-1)} = \frac{\partial h_1}{\partial w_1} \left(h_1^{\eta(\nu-1)} \left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu} \right) \right) dw_1 \wedge dw_2,$$

whose coefficient is also a subelliptic multiplier with

$$\left\| \left\| \frac{\partial h_1}{\partial w_1} \left(h_1^{\eta(\nu-1)} \left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu} \right) \right) \phi \right\| \right\|_{\frac{1}{2^{\nu \cdot 4}(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

10.2.3. Since \tilde{h}_2 is also a subelliptic multiplier,

$$\left\| \left\| \tilde{h}_2 \left(h_1^{\eta(\nu-1)} \left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu} \right) \right) \phi \right\| \right\|_{\frac{1}{2^\nu \cdot 4(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

10.2.4. Combining the inequalities in the last two paragraphs, and using the fact that $|h_1^\eta| \lesssim |\partial_{z_1} h_1| + |\tilde{h}_2|$,

$$\begin{aligned} & \left\| \left\| h_2^{(\nu)} \phi \right\| \right\|_{\frac{1}{2^\nu \cdot 4(4s^2-1)}}^2 \\ &= \left\| \left\| h_1^{\nu\eta} \left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu} \right) \phi \right\| \right\|_{\frac{1}{2^\nu \cdot 4(4s^2-1)}}^2 \\ &= \left\| \left\| h_1^\eta \left(h_1^{\eta(\nu-1)} \left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu} \right) \right) \phi \right\| \right\|_{\frac{1}{2^\nu \cdot 4(4s^2-1)}}^2 \\ &\lesssim \left\| \left\| \frac{\partial h_1}{\partial w_1} \left(h_1^{\eta(\nu-1)} \left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu} \right) \right) \phi \right\| \right\|_{\frac{1}{2^\nu \cdot 4(4s^2-1)}}^2 \\ &\quad + \left\| \left\| \tilde{h}_2 \left(h_1^{\eta(\nu-1)} \left(\frac{\lambda!}{(\lambda-\nu)!} w_2^{\lambda-\nu} + \sum_{j=\nu}^{\lambda-1} \frac{j!}{(j-\nu)!} a_j(h_1) w_2^{j-\nu} \right) \right) \phi \right\| \right\|_{\frac{1}{2^\nu \cdot 4(4s^2-1)}}^2 \\ &\lesssim Q(\phi, \phi). \end{aligned}$$

This finishes the induction process.

10.2.5. Setting $\nu = \lambda$, we get

$$\left\| \left\| h_2^{(\lambda)} \phi \right\| \right\|_{\frac{1}{2^\lambda \cdot 4(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

But $h_2^{(\lambda)} = h_1^{\eta\lambda} \lambda!$, therefore

$$\left\| \left\| h_1^{\lambda\eta} \phi \right\| \right\|_{\frac{1}{2^\lambda \cdot 4(4s^2-1)}}^2 \lesssim Q(\phi, \phi).$$

10.3. Siu's method: Conclusion and End of Calculation.

10.3.1. Since

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \langle h_1, \tilde{h}_2 \rangle \leq (4s^2 - 1)s,$$

by Proposition 4.17,

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \langle h_1^{\lambda\eta}, \tilde{h}_2 \rangle = \eta\lambda \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0} / \langle h_1, \tilde{h}_2 \rangle \leq (4s^2 - 1)s\eta\lambda.$$

For $i = 1, 2$, by the Lemma 4.12,

$$w_i^{(4s^2-1)s\lambda\eta} \in \langle h_1^{\lambda\eta}, \tilde{h}_2 \rangle.$$

Thus $w_i^{(4s^2-1)s\lambda\eta}$ is also a multiplier with

$$|w_i^{(4s^2-1)s\lambda\eta}| \lesssim |h_1^{\eta\lambda}| + |\tilde{h}_2|.$$

Hence

$$\begin{aligned}
& \left\| \left\| w_i^{(4s^2-1)s\lambda\eta} \phi \right\| \right\|_{\frac{1}{2^\lambda \cdot 4(4s^2-1)}}^2 \\
& \lesssim \left\| \left\| h_1^{\eta\lambda} \phi \right\| \right\|_{\frac{1}{2^\lambda \cdot 4(4s^2-1)}}^2 + \left\| \left\| \tilde{h}_2 \phi \right\| \right\|_{\frac{1}{2^\lambda \cdot 4(4s^2-1)}}^2 \\
& \lesssim \left\| \left\| h_1^{\eta\lambda} \phi \right\| \right\|_{\frac{1}{2^\lambda \cdot 4(4s^2-1)}}^2 + \left\| \left\| \tilde{h}_2 \phi \right\| \right\|_{\frac{1}{4(4s^2-1)}}^2 \lesssim Q(\phi, \phi).
\end{aligned}$$

By radical property of subelliptic multipliers Proposition 3.2(i), one has for each $i = 1, 2$ that

$$\left\| \left\| w_i \phi \right\| \right\|_{\frac{1}{2^\lambda \cdot 4s\eta\lambda(4s^2-1)^2}}^2 \lesssim Q(\phi, \phi).$$

Taking the Jacobian, one obtains by Propositions 3.2(ii) and 3.2(iii) that

$$\left\| \left\| \phi \right\| \right\|_{\frac{1}{2^{\lambda+1} \cdot 4s\eta\lambda(4s^2-1)^2}}^2 \lesssim Q(\phi, \phi),$$

and this terminates the calculation.

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